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ORIGINAL RESEARCH ARTICLE

The role of thermal effusivity on the incipient growth of the surface temperature in a semi-infinite region absorbing heat flux at the surface

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ABSTRACT

Heat conduction theory stipulates that two thermo-physical properties of materials: the thermal conductivity "k" and the thermal diffusivity "α" influence the temperature evolution in regular and irregular bodies as a response to various cooling/heating conditions. The traditional statement involving the two thermo-physical properties is examined at length in the present study for the case of a semi-infinite region. The primary objective of the present study is to investigate the influence of the less known thermo-physical property called the thermal effusivity "e" on the incipient surface temperature rise in a semi-infinite body affected by uniform surface heat flux. The secondary objective of the study is to identify a key figure of merit named the dimensionless threshold time that separates the incipient temperature elevation in a semi-infinite region from the incipient temperature elevation in a large wall of finite thickness under the same uniform surface heat flux. The outcome of the methodical analysis suggests that the accurate estimate for the dimensionless threshold time τ_{th} in the semi-infinite region should be 0.10.

Keywords: semi-infinite region; uniform surface heat flux; incipient surface temperature elevation; role of the thermal effusivity

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1. Introduction

Unsteady heat conduction in regular and irregular bodies has remarkable importance in a multitude of engineering scenarios around the world, as documented in the heat conduction textbooks by Carslaw and Jaeger[1], Arpaci[2], Luikov[3], Myers[4], Grigull and Sandner[5], Özişik[6], Poulikakos[7], Jiji[8] and Kakaç et al.[9].

With regards to unsteady heat conduction in regular and irregular bodies subject to various heating/cooling conditions, the plethora of mathematical procedures has led to an assortment of 1) exact analytical solutions, 2) approximate analytical solutions, and 3) approximate/numerical solutions.

From the framework of thermo-physical properties of materials, it transcends that the temperature distribution in regular and irregular bodies is dependent upon two thermo-physical properties: the thermal conductivity "k" and the thermal diffusivity "α"[1–9].

The present study focuses on a certain regular body, namely the semi-infinite region with constant initial temperature and temperature-invariant thermo-physical properties being heated with uniform heat flux at the surface (Neumann boundary condition).

The primary objective of the study is to investigate the potential interplay between the less known thermo-physical property called the thermal effusivity "e" and the surface temperature growth in the semi-infinite region as a consequence of continuous heating.

The secondary objective of the study revolves around the identification of a dimensionless threshold time τ_{th} that separates a semi-infinite region from a large wall with finite thickness during the initial stages of the continuous heating of both. This part establishes a key figure of merit that is valuable in the analysis of this kind of unsteady heat conduction problem.

2. Semi-infinite region affected by uniform heat flux

Consider a semi-infinite region exposed to a nearby heat source. Uniform heat flux q_{s} emanates from the heat source in the form of electrical heating[10] or radiative heating[11] and subsequently intrudes into the semi-infinite region.

Under the assumption that a material has constant thermal conductivity "k" and constant thermal diffusivity "α" in the temperature interval of operation $\Delta T = |T_{\text{in}} - T_{\text{fin}}|$, the temperature distribution in a semi-infinite region available in heat conduction textbooks[1–9] is

$$T_{\text{sir}}(y,t) = T_{\text{in}} + \frac{q_{\text{s}}}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{y^2}{4\alpha t}\right) - y\ \text{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right)\right] \tag{1}$$

where the space coordinate y is measured from the free surface $y = 0$ of the semi-infinite region inward. Herein, the subscript "sir" attached to the symbol T denotes a semi-infinite region.

Upon inspection of Equation (1), it is ascertainable that the temperature distribution $T_{\text{sir}}(y,t)$ depends on the thermal conductivity "k" in the ratio $\frac{q_{\text{s}}}{k}$ preceding the bracket and on the thermal diffusivity "α" in three different places inside the bracket. They are 1) in the square root, 2) in the argument of the exponential function, and 3) in the argument of the complimentary error function.

Primary target temperatures

Specifically, the primary target temperature in the continuous heating of the semi-infinite region under study here turns out to be the surface temperature T_{s}. This is so because T_{s} is the highest temperature in the semi-infinite region at any time during the heating, and it must remain below the melting temperature T_{melt} of the material. A list of typical melting temperatures T_{melt} of metals and alloys is available[12].

First, evaluating Equation (1) at the surface $y = 0$ in the semi-infinite region indicates that erfc (0) is nullified and exp (0) equates to 1. Correspondingly, the surface temperature distribution becomes

$$T_{\text{s}}(t) = T_{\text{s,sir}}(0,t) = T_{\text{in}} + \frac{q_{\text{s}}}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\ (0) - 0\right] \tag{2}$$

where the subscript "s" attached to T indicates surface. Here, the thermal conductivity "k" appears in the ratio $\frac{q_{\text{s}}}{k}$ preceding the bracket, whereas the thermal diffusivity "α" appears in the square root inside the bracket.

Second, replacing "α" by its three components k, ρ, c_{v}, Equation (2) readily simplifies to

$$T_{\text{s}}(t) = T_{\text{in}} + \frac{2}{\sqrt{\pi}}\frac{q_{\text{s}}}{k}\sqrt{\frac{k}{\rho c_{\text{v}}}}\sqrt{t} \tag{3}$$

Third, regrouping the three components k,ρ,c_{v} gives way to the definitive equation for the surface temperature distribution.

$$T_{\text{s}}(t) = T_{\text{in}} + \frac{2}{\sqrt{\pi}}\frac{q_{\text{s}}}{\sqrt{k\rho c_{\text{v}}}}\sqrt{t} \tag{4}$$

3. The thermal effusivity

The thermal effusivity "e", also called the heat penetration coefficient or the thermal responsivity, is a thermo-physical property outlined in the heat conduction textbook by Grigull and Sandner[5]. Surprisingly, the thermal effusivity "e" is not mentioned in the other six heat conduction textbooks[1–4,6–9] cited in the Introduction.

The thermal effusivity "e" is defined as the square root of the product of the thermal conductivity "k" and the heat capacity at constant volume "C_{v}":

$$e = \sqrt{kC_{\mathrm{v}}} = \sqrt{k\rho\, c_{\mathrm{v}}} \tag{5}$$

The earliest use of the term thermal effusivity "e" traceable with modern scientific indexing tools was done by Krischer and Esdorn[13] in 1955.

The thermal effusivity "e" seems like a contrived thermo physical property, but it does have real meaning. If two separate semi-infinite regions 1 and 2 at different temperatures, T_1 and T_2, and with different thermal effusivities, "e_1" and "e_2", suddenly come into perfect contact, then the surface in each semi-infinite region will quickly reach an interface temperature.

$$T_{\mathrm{if}} = \frac{T_1 e_1 + T_2 e_2}{e_1 + e_2} \tag{6}$$

This relation constitutes a weighted mean of the thermal effusivities "e_1" and "e_2". It transcends the fact that they will be closer to the temperature of the semi-infinite region with higher effusivity. In other words, the three options are: if "e_1" = "e_2", then T_{if} lies halfway between T_1 and T_2, if "e_1" > "e_2", then T_{if} will be closer to T_1, and if "e_1" < "e_2", then T_{if} will be closer to T_2. In this context, the thermal effusivity "e" emerges as an important metric for textiles and fabrics. Additional information is given in Xin and Tao[14].

Conversely, the thermal diffusivity "α" refers to the thermal conductivity "k" divided by the heat capacity at constant volume "C_{v}".

$$\alpha = \frac{k}{C_v} = \frac{k}{\rho c_v} \tag{7}$$

At this point, it is important to realize that while the expression for "e" in Equation (5) and the expression for "α" in Equation (7) contain the same physical quantities k, ρ, c_{v}, their contributions to unsteady heat conduction are quite different. On one hand, the thermal diffusivity α is associated with the speed at which thermal equilibrium in a material can be reached[1–9]. On the other hand, the thermal effusivity "e" is connected to the capacity of a material to absorb heat[5].

Finally, the interplay between the thermal effusivity "e" and the thermal diffusivity "α" can be visualized explicitly by rewriting Equation (5) in the following manner:

$$e = \sqrt{k\rho\, c_{\mathrm{v}}} = \frac{k}{\sqrt{\alpha}} = \rho\, c_{\mathrm{v}}\sqrt{\alpha} \tag{8}$$

Next, returning to Equation (4) combined with Equation (5) brings forth the definitive equation for the surface temperature distribution in the semi-infinite region receiving uniform surface heat flux q_{s} continually. That is,

$$T_{\mathrm{s}}(t) = T_{\mathrm{in}} + \frac{2}{\sqrt{\pi}}\frac{q_{\mathrm{s}}}{e}\sqrt{t} \tag{9}$$

Note that in the second term of the equation, T_{s} is directly proportional to q_{s} and $\sqrt{t}$, while it is inversely proportional to "e".

An extensive list of thermal effusivities "e" for engineering materials was compiled by Baines[15]. Here, the typical values of "e" were grouped in the following manner: 1) 0.01–0.4 for insulating materials, 2) 0.4–1.5

kW m^{-2} K^{-1} s$^{-1/2}$ for polymers, 3) 1.5–9.0 kW m^{-2} K^{-1} s$^{-1/2}$ for ceramics; and 4) 7–37 kW m^{-2} K^{-1} s$^{-1/2}$ for metals. Another list of thermal effusivities "e" for materials related to electronics cooling was compiled by Lasance[16].

4. Identification of the threshold time between a semi-infinite region and a large wall of finite thickness

The temperature distribution in a semi-infinite region subject to uniform surface heat flux copied from Equation (1) is

$$T_{\mathrm{sir}}(y,t) = T_{\mathrm{in}} + \frac{q_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{y^2}{4\alpha t}\right) - y\,\mathrm{erfc}\left(\frac{y}{\sqrt{4\alpha t}}\right)\right] \tag{10}$$

From geometry, the surface $y = 0$ in the semi-infinite region is coincident with the surface $x = L$ on the right side of a large wall of finite thickness $2L$. Elaborating on this issue, we proceed to evaluate the interior temperature $T_{\mathrm{sir}}(L,t)$ in the semi-infinite region coinciding with the mid-plane temperature $T_{\mathrm{mp}}(0.t)$ in the large wall of finite thickness $2L$. That is

$$T_{\mathrm{mp}}(0.t) = T_{\mathrm{sir}}(L,t) = T_{\mathrm{in}} + \frac{q_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{L^2}{4\alpha t}\right) - L\,\mathrm{erfc}\left(\frac{L}{\sqrt{4\alpha t}}\right)\right] \tag{11}$$

where the subscript "mp" annexed to the symbol T stands for mid-plane in the large wall of finite thickness $2L$.

Dividing Equation(11) by $\frac{q_s L}{k}$ and grouping the temperature difference $T_{\mathrm{mp}}(t) - T_{\mathrm{in}}$, the resulting equation can be rewritten as

$$\frac{1}{\frac{q_s L}{k}}\left[T_{\mathrm{mp}}(t) - T_{\mathrm{in}}\right] = \sqrt{\frac{4}{\pi}}\sqrt{\frac{\alpha t}{L^2}}\exp\left(-\frac{L^2}{4\alpha t}\right) - \mathrm{erfc}\left(\frac{L}{2\sqrt{\alpha t}}\right) \tag{12}$$

From here, manipulating the half-thickness L of the large wall on the right-hand side in Equation (12), the equation switches adequately to

$$\frac{1}{\frac{q_s L}{k}}\left[T_{\mathrm{mp}}(t) - T_{\mathrm{in}}\right] = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\alpha t}{L^2}}\exp\left(-\frac{1}{4\left(\frac{\alpha t}{L^2}\right)}\right) - \mathrm{erfc}\left(\frac{1}{2\sqrt{\frac{\alpha t}{L^2}}}\right) \tag{13}$$

In the previous equation, the ratio $\frac{\alpha t}{L^2}$ appears in three places: the square root, the argument of the exponential function, and the argument of the complimentary error function. Since the ratio $\frac{\alpha t}{L^2}$ defines the dimensionless time τ, Equation (13) can be further simplified to the working equation.

$$\phi_{mp}(\tau) = \frac{2}{\sqrt{\pi}}\sqrt{\tau}\exp\left(-\frac{1}{4\tau}\right) - \mathrm{erfc}\left(\frac{1}{2\sqrt{\tau}}\right) \tag{14}$$

where ϕ_{mp} denotes the dimensionless mid-plane temperature $\frac{T_{\mathrm{mp}}(t)-T_{\mathrm{in}}}{\frac{q_s L}{k}}$ in the large wall of half-thickness L.

Now, we proceed to estimate the dimensionless threshold time τ_{th}, a figure-of-merit that separates the dimensionless temperature sub-distribution in the semi-infinite region from the dimensionless temperature distribution in a large wall of half-thickness L. First, to estimate the dimensionless mid-plane temperature ϕ_{mp} at $\tau = 0$, a simple trial-and-error procedure is pursued. Various values of the dimensionless time τ ranging between $\tau = 0$ and 0.10 are tested in Equation (14). In particular, when the dimensionless threshold time $\tau = 0.10$ is substituted in Equation(14), the dimensionless mid-plane temperature ϕ_{mp} amounts to

$$\phi_{\mathrm{mp}}(0.10) = 0.35683 \times 0.08209 - 0.02535 = 0.0394 \approx 0.005 \tag{15}$$

Alternatively, this finding may be rephrased in the following manner: setting a very small 0.5% error in the dimensionless mid-plane temperature $\phi_{\mathrm{mp}}(0)$ for the large wall of finite half-thickness L, the correlative dimensionless threshold time that results is $\tau_{\mathrm{th}} = 0.10$. Moreover, the validity of this number can be confirmed by looking at the right lower part of **Figure 1**.

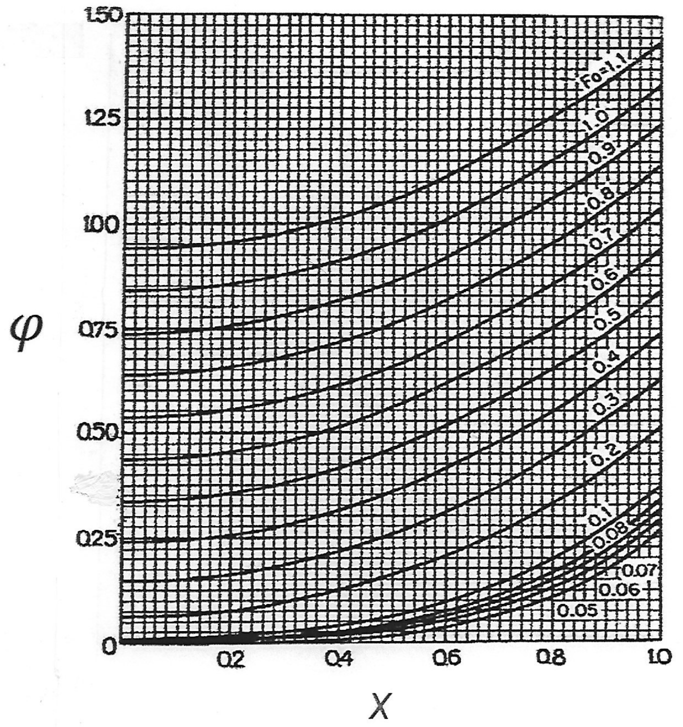

Figure 1. Dimensionless temperature distribution in the right part of a large wall of finite thickness 2L receiving uniform heat flux.

5. Conclusion

The main conclusion that may be drawn from the present study is that the thermal effusivity "e" plays a significant role in the incipient growth of the surface temperature in a semi-infinite region under the combination of constant initial temperature and uniform surface heat flux. The material of the semi-infinite region has temperature—invariant thermal conductivity "k" and temperature—invariant thermal diffusivity "α".

It was found that the impact of the thermal effusivity "e" on the surface temperature growth in a semi-infinite region is dominant between the dimensionless initial time $\tau = 0$ and the dimensionless threshold time $\tau_{\mathrm{th}} = 0.10$. At other interior locations beyond $y > 0$ different than the surface $y = 0$, the correlative dimensionless temperature vs. time relation in Equation (1) is affected by the thermal conductivity "k" outside the bracket and by the thermal diffusivity "α" inside the bracket.

Conflict of interest

The author declares no conflict of interest.

Nomenclature

c_{v}	Specific heat capacity at constant volume
C_{v}	Heat capacity at constant volume, ρc_{v}
e	Thermal effusivity, $\sqrt{k\rho c_{\mathrm{v}}}$
Fo	Fourier modulus in Figure 1, $\frac{t}{L^2/\alpha}$

k	Thermal conductivity
$2L$	Thickness of large wall
q_s	Uniform surface heat flux
t	Time
t_{th}	Threshold time
T	Temperature
T_{in}	Initial temperature
T_s	Surface temperature
x	Space coordinate in a large wall
X	Dimensionless x in Figure 1, $\frac{x}{L}$
y	Space coordinate in a semi-infinite region

Greek letters

α	Thermal diffusivity, $\frac{k}{\rho c_v}$
ρ	Density
τ	Dimensionless time, $\frac{t}{L^2/\alpha}$
τ_{th}	Dimensionless threshold time, $\frac{t_{th}}{L^2/\alpha}$
ϕ	Dimensionless T, $\frac{T-T_{in}}{q_s/k}$

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