

Anisotropic Behaviour and Elastic Moduli of Average Isotropic and Anisotropic Elastic Media

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ABSTRACT

Using matricant method elastic moduli of occasionally heterogeneous isotropic and anisotropic elastic media were received. Anisotropic behaviour and conditions for change in anisotropy of media associated with averaging of one-dimensional periodic structures was determined.

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1. Introduction

Inhomogeneity and anisotropy is the most widespread behaviour of real media. When studying wave process in Earth crust and in composite mechanics a model of homogeneous or layered media with average characteristics is widely used as a real media model. Currently because of intensive development and application of non-destructive methods, sound velocity methods, tomography, remote sensing, *etc.* similar models are used in various fields of science. In case of assumption of heterogeneous medium with homogeneous model application of the most adequate model of primary medium is important. This work sets the results of investigation of anisotropic behaviour of homogeneous models of heterogeneous isotropic and anisotropic elastic media. Based on matricant method^[1-3] the formulae determining values of elastic media were received. Research method determines anisotropy of averaged media.

2. Elastic models of averaged occasionally heterogeneous isotropic medium

Within the frames of matricant method and equivalent system of ordinary differential first-order equations describing propagation of elastic harmonic wave in heterogeneous isotropic medium in case of inhomogeneity along axis z is written as:

$$\frac{d\vec{W}}{dz} = B(z) \vec{W}; \vec{W} = (U_z, \sigma_{zz}, U_x, \sigma_{xz}, U_y, \sigma_{yz})^t. \quad (1)$$

Matrix of coefficients $B(z)$ is structured as:

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & b_{15} & 0 \\ b_{21} & 0 & 0 & b_{24} & 0 & b_{26} \\ b_{24} & 0 & 0 & b_{34} & 0 & 0 \\ 0 & b_{13} & b_{43} & 0 & b_{45} & 0 \\ b_{26} & 0 & 0 & 0 & 0 & b_{56} \\ 0 & b_{15} & b_{45} & 0 & b_{65} & 0 \end{pmatrix} \quad (2)$$

Elements $b_{ij}(z)$:

$$\begin{aligned} b_{11} &= \frac{1}{c_{11}}; b_{13} = ik_x \frac{c_{12}}{c_{11}}; b_{15} = ik_y \frac{c_{12}}{c_{11}}; \\ b_{21} &= -\omega^2 \rho; b_{24} = ik_x; b_{26} = ik_y; \\ b_{34} &= \frac{1}{c_{66}}; b_{43} = -\omega^2 \rho + k_x^2 \left(c_{11} - \frac{c_{12}^2}{c_{11}} \right) + k_y^2 c_{66}; \\ b_{45} &= k_x k_y \left(c_{66} + c_{12} - \frac{c_{12}^2}{c_{11}} \right); b_{56} = \frac{1}{c_{66}}; \\ b_{65} &= -\omega^2 \rho + k_x^2 c_{66} + k_y^2 \left(c_{11} - \frac{c_{12}^2}{c_{11}} \right). \end{aligned} \quad (3)$$

In case of isotropic media:

$$\begin{aligned} [c_{11} = c_{22} = c_{33} = \lambda + 2\mu = c_{11}; \\ c_{11} = c_{22} = c_{33} = \lambda + 2\mu = c_{11}; \end{aligned} \quad (4)$$

$$c_{44} = c_{55} = c_{66} = \mu = \frac{1}{2} (c_{11} - c_{12}) = c_{66}.$$

Given periodic inhomogeneity of medium:

$$\begin{aligned} \lambda(z) &= \lambda(z+h); \mu(z) = \mu(z+h); \\ c_{ij}(z) &= c_{ij}(\lambda_0 + h), \end{aligned} \quad (5)$$

where h is inhomogeneity period.

If wave length is $\lambda_0 \gg h$, occasionally heterogeneous medium can be considered as homogeneous medium described by matrix of coefficients as averaged based on inhomogeneity period:

$$\frac{d\vec{W}}{dz} = \langle B \rangle \vec{W}; \langle B \rangle = \frac{1}{h} \int_0^h B(z) dz. \quad (6)$$

From (6) it follows:

$$\begin{aligned} \langle b_{ij} \rangle &= \frac{1}{h} \int_0^h b_{ij}(z) dz; \\ \langle b_{11} \rangle &= \left\langle \frac{1}{c_{11}} \right\rangle; \langle b_{13} \rangle = ik_x \left\langle \frac{c_{12}}{c_{11}} \right\rangle \\ \langle b_{15} \rangle &= ik_y \left\langle \frac{c_{12}}{c_{11}} \right\rangle; \langle b_{34} \rangle = \left\langle \frac{1}{c_{66}} \right\rangle; \\ \langle b_{21} \rangle &= -\omega^2 \langle \rho \rangle; \langle b_{24} \rangle = ik_x; \langle b_{26} \rangle = ik_y; \\ \langle b_{43} \rangle &= -\omega^2 \langle \rho \rangle + k_x^2 \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + k_y^2 \langle c_{66} \rangle; \\ \langle b_{45} \rangle &= k_x k_y \left\langle c_{66} + c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle; \langle b_{56} \rangle = \left\langle \frac{1}{c_{66}} \right\rangle; \\ \langle b_{65} \rangle &= -\omega^2 \langle \rho \rangle + k_x^2 \langle c_{66} \rangle + k_y^2 \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle. \end{aligned} \quad (7)$$

Similar system of first-order equations for homogeneous elastic medium of hexagonal anisotropy (axis $z \parallel A_6$ is hexagonal axis) is written as:

$$\frac{d\vec{W}}{dz} = B\vec{W}; \vec{W} = (U_z, \sigma_{zz}, U_x, \sigma_{xz}, U_y, \sigma_{yz})^t \quad (8)$$

Structure of matrix of coefficients B in (8) is the same as (2). Elements b_{ij} are written as:

$$\begin{aligned} b_{11} &= \frac{1}{c_{22}}; b_{12} = ik_x \frac{c_{13}^0}{c_{33}^0}; b_{15} = ik_y \frac{c_{13}^0}{c_{33}^0}; \\ b_{21} &= -\omega^2 \rho; b_{24} = ik_x; b_{26} = ik_y; \\ b_{34} &= \frac{1}{c_{44}}; b_{43} = -\omega^2 \rho + k_x^2 \left(c_{11}^0 - \frac{c_{13}^0{}^2}{c_{33}^0} \right) + k_y^2 c_{66}^0; \\ b_{45} &= k_x k_y \left(c_{66}^0 + c_{12}^0 - \frac{c_{13}^0{}^2}{c_{33}^0} \right) \\ b_{56} &= \frac{1}{c_{44}}; b_{65} = -\omega^2 \rho_0 + k_x^2 c_{66}^0 + k_y^2 \left(c_{11}^0 - \frac{c_{13}^0{}^2}{c_{33}^0} \right) \end{aligned} \quad (9)$$

Comparison of elements of averaged matrix (7) and matrix of coefficients of homogeneous anisotropic medium of hexagonal symmetry (9) ends in relations:

$$\left\langle \frac{1}{c_{11}} \right\rangle = \frac{1}{c_{33}^0}; \left\langle \frac{1}{c_{66}} \right\rangle = \frac{1}{c_{44}^0}; \langle c_{66} \rangle = c_{66}^0;$$

$$\begin{aligned} \langle \rho \rangle &= \rho_0; \left\langle \frac{c_{12}}{c_{11}} \right\rangle = \frac{c_{13}^0}{c_{33}^0}; \left\langle c_{11} - \frac{c_{12}^2}{c_{33}} \right\rangle = c_{11}^0 - \frac{c_{13}^0{}^2}{c_{33}^0}; \\ \left\langle c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle &= c_{12}^0 - \frac{c_{13}^0{}^2}{c_{33}^0}. \end{aligned} \quad (10)$$

From (10) it follows:

$$\begin{aligned} c_{33}^0 &= \left\langle \frac{1}{c_{11}} \right\rangle^{-1}; c_{44}^0 = \left\langle \frac{1}{c_{66}} \right\rangle^{-1}; c_{66}^0 = \langle c_{66} \rangle; \\ c_{13}^0 &= \left\langle \frac{c_{12}}{c_{11}} \right\rangle \left\langle \frac{1}{c_{11}} \right\rangle^{-1} = \left\langle \frac{c_{12}}{c_{11}} \right\rangle c_{33}^0; \\ c_{11}^0 &= \langle c_{11} \rangle - \left\langle \frac{c_{12}^2}{c_{11}} \right\rangle + \frac{c_{13}^0{}^2}{c_{33}^0} \\ c_{12}^0 &= \langle c_{12} \rangle - \left\langle \frac{c_{12}^2}{c_{11}} \right\rangle + \frac{c_{13}^0{}^2}{c_{33}^0}; c_{66}^0 = \frac{1}{2} (c_{11}^0 - c_{12}^0) \end{aligned} \quad (11)$$

Elastic modules c_{ii}^0 match up with elastic modules of hexagonal anisotropy.

Thus, an averaged heterogeneous isotropic medium is equivalent to homogeneous elastic medium of hexagonal anisotropy with elastic modules:

$$\begin{aligned} c_{33}^0 &= \left\langle \frac{1}{c_{11}} \right\rangle^{-1} = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle^{-1}; c_{44}^0 = \left\langle \frac{1}{c_{44}} \right\rangle^{-1} = \left\langle \frac{1}{\mu} \right\rangle^{-1}; \\ \langle \rho \rangle &= \rho_0; c_{66}^0 = \frac{1}{2} (c_{11}^0 - c_{12}^0) = \langle \mu \rangle; \\ c_{11}^0 &= \langle \lambda + 2\mu \rangle - \left\langle \frac{\lambda^2}{\lambda + 2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2 \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^{-1}; \\ c_{12}^0 &= \langle \lambda \rangle - \left\langle \frac{\lambda^2}{\lambda + 2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2 \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^{-1}; \\ c_{13}^0 &= \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle \left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^{-1}; \\ c_{66}^0 &= \frac{1}{2} (c_{11}^0 - c_{12}^0) = \frac{1}{2} ((\lambda + 2\mu) - \langle \lambda \rangle) = \langle \mu \rangle. \end{aligned} \quad (12)$$

Averaging of elastic medium of hexagonal anisotropy occasionally heterogeneous along axis $z(z \parallel A_6)$ leaves anisotropic behaviour unaffected.

From averaging of matrix of coefficients B (9) under its occasional dependence:

$$b_{ij}(z) = b_{ij}(z + h)$$

it follows:

$$\begin{aligned} c_{33}^0 &= \left\langle \frac{1}{c_{33}} \right\rangle; c_{44}^0 = \left\langle \frac{1}{c_{44}} \right\rangle^{-1}; \\ c_{66}^0 &= \langle c_{66} \rangle \langle \rho \rangle = \rho_0; c_{13}^0 = \left\langle \frac{c_{13}}{c_{33}} \right\rangle \left\langle \frac{1}{c_{33}} \right\rangle^{-1}; \end{aligned}$$

$$c_{12}^0 = \langle c_{12} \rangle - \left\langle \frac{c_{13}^2}{c_{33}} \right\rangle + \frac{c_{13}^0{}^2}{c_{33}}; c_{11}^0 = \langle c_{11} \rangle - \left\langle \frac{c_{13}^2}{c_{33}} \right\rangle + \frac{c_{13}^0{}^2}{c_{33}};$$

$$c_{66}^0 = \langle c_{66} \rangle = \frac{1}{2} \langle c_{11} \rangle - \frac{1}{2} \langle c_{12} \rangle \quad (14)$$

From now on c_{ij} is elastic models of heterogeneous medium; c_{ij}^0 – elastic models of homogeneous averaged medium.

Conclusion: inhomogeneity of anisotropic medium of hexagonal symmetry along axis z , parallel hexagonal axis A_6 , leaves its anisotropic behaviour unaffected.

Averaged heterogeneous anisotropic medium of hexagonal symmetry along hexagonal axis has anisotropic behaviour of hexagonal symmetry.

Anisotropic behaviour remains unchanged.

3. Elastic modules and anisotropic behaviour of occasionally heterogeneous media of hexagonal symmetries averaged along axis $x \perp A_6$

Within the frames of matricant method wave process in anisotropic media of hexagonal symmetry heterogeneous along axis x are described by the system of ordinary differential first-order equations:

$$\frac{d\vec{W}}{dx} = B(x)\vec{W}; \vec{W} = (U_x, \sigma_{xx}, U_y, \sigma_{yx}, U_z, \sigma_{xz})^t. \quad (15)$$

Matrix of coefficients $B(x)$ is structured as (2) with elements $b_{ij}(x)$:

$$b_{12} = \frac{1}{c_{11}}; b_{13} = ik_y \frac{c_{12}}{c_{11}}; b_{15} = ik_z \frac{c_{13}}{c_{11}};$$

$$b_{21} = -\omega^2 \rho; b_{24} = ik_y; b_{26} = ik_z; b_{34} = \frac{1}{c_{66}};$$

$$b_{43} = -\omega^2 \rho + k_z^2 c_{44} + k_y^2 \left(c_{11} - \frac{c_{12}^2}{c_{11}} \right);$$

$$b_{45} = k_y k_z \left(c_{44} + c_{13} - \frac{c_{12} c_{13}}{c_{11}} \right); \quad (16)$$

$$b_{56} = \frac{1}{c_{44}}; b_{65} = -\omega^2 \rho + k_y^2 c_{44} + k_z^2 \left(c_{33} - \frac{c_{13}^2}{c_{11}} \right)$$

In case of occasional inhomogeneity along x :

$$\rho(x) = \rho(x+h);; c_{ij}(x) = c_{ij}(x+h). \quad (17)$$

Assuming that $\lambda_0 \gg h$ (λ_0 is wave length, h is inhomogeneity period) from

$$\langle B(x) \rangle = \frac{1}{h} \int_0^h B(x) dx \quad (18)$$

elements of averaged matrix of coefficient follow:

$$\langle b_{12} \rangle = \left\langle \frac{1}{c_{11}} \right\rangle; b_{13} = ik_y \left\langle \frac{c_{12}}{c_{11}} \right\rangle; b_{15} = ik_z \left\langle \frac{c_{13}}{c_{11}} \right\rangle;$$

$$\langle b_{21} \rangle = -\omega^2 \langle \rho \rangle; \langle b_{34} \rangle = \left\langle \frac{1}{c_{66}} \right\rangle;$$

$$\langle b_{43} \rangle = -\omega^2 \langle \rho \rangle + k_y^2 \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + k_z^2 \langle c_{44} \rangle;$$

$$\langle b_{45} \rangle = k_y k_z \left\langle c_{44} + c_{13} - \frac{c_{12} c_{13}}{c_{11}} \right\rangle; \quad (19)$$

$$\langle b_{56} \rangle = \left\langle \frac{1}{c_{44}} \right\rangle; \langle b_{65} \rangle = -\omega^2 \langle \rho \rangle + k_y^2 \langle c_{44} \rangle + k_z^2 \left\langle c_{33} - \frac{c_{13}^2}{c_{11}} \right\rangle$$

Set of equations for anisotropic media of rhombic symmetry is written as (15) with the structure of matrix of coefficients B as (2).

Elements of matrix B for media of rhombic symmetry are written as:

$$b_{12} = \frac{1}{c_{11}^0}; b_{13} = ik_y \frac{c_{12}^0}{c_{11}^0}; b_{15} = ik_z \frac{c_{13}^0}{c_{11}^0}; b_{21} = -\omega^2 \rho_0;$$

$$b_{24} = ik_y; b_{26} = ik_z; b_{34} = \frac{1}{c_{66}^0};$$

$$b_{43} = -\omega^2 \rho_0 + k_z^2 c_{44}^0 + k_y^2 \left(c_{22} - \frac{c_{12}^0{}^2}{c_{11}^0} \right);$$

$$b_{45} = k_y k_z \left(c_{44}^0 + c_{23}^0 - \frac{c_{12}^0 c_{13}^0}{c_{11}^0} \right); \quad (20)$$

$$b_{56} = \frac{1}{c_{55}^0}; b_{65} = -\omega^2 \rho_0 + k_y^2 c_{44}^0 + k_z^2 \left(c_{33}^0 - \frac{c_{13}^0{}^2}{c_{11}^0} \right)$$

From comparison of b_{ij} (19) and (20) we obtain:

$$\left\langle \frac{1}{c_{11}} \right\rangle = \frac{1}{c_{11}^0}; \left\langle \frac{c_{12}}{c_{11}} \right\rangle = \frac{c_{12}^0}{c_{11}^0}; \left\langle \frac{c_{13}}{c_{11}} \right\rangle = \frac{c_{13}^0}{c_{11}^0};$$

$$\langle \rho \rangle = \rho_0; \langle c_{44} \rangle = c_{44}^0; \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle = c_{22}^0 - \frac{c_{12}^0{}^2}{c_{11}^0};$$

$$\left\langle c_{13} - \frac{c_{12} c_{13}}{c_{11}} \right\rangle = c_{23}^0 - \frac{c_{12}^0 c_{13}^0}{c_{11}^0}; \left\langle \frac{1}{c_{44}} \right\rangle = \frac{1}{c_{55}^0}. \quad (21)$$

From (21) it follows:

$$c_{11}^0 = \left\langle \frac{1}{c_{11}} \right\rangle^{-1}; c_{12}^0 = \left\langle \frac{c_{12}}{c_{11}} \right\rangle c_{11}^0;$$

$$\begin{aligned}
c_{13}^0 &= \left\langle \frac{c_{13}}{c_{11}} \right\rangle c_{11}^0; c_{44}^0 = \langle c_{44} \rangle; \\
c_{55}^0 &= \left\langle \frac{1}{c_{44}} \right\rangle^{-1}; c_{66}^0 = \left\langle \frac{1}{c_{66}} \right\rangle^{-1}; \\
c_{22}^0 &= \langle c_{11} \rangle - \left\langle \frac{c_{12}^2}{c_{11}} \right\rangle + \frac{c_{12}^0{}^2}{c_{11}^0}; \\
c_{23}^0 &= \langle c_{13} \rangle - \left\langle \frac{c_{12}c_{13}}{c_{11}} \right\rangle + \frac{c_{12}^0 c_{13}^0}{c_{11}^0}; \\
c_{33}^0 &= \langle c_{33} \rangle - \left\langle \frac{c_{13}^2}{c_{11}} \right\rangle + \frac{c_{13}^0{}^2}{c_{11}^0}.
\end{aligned} \tag{22}$$

We obtained nine independent elastic modules derived from averaging of matrix of coefficients $B(x)$ along x :

$$c_{11}^0, c_{22}^0, c_{33}^0, c_{44}^0, c_{55}^0, c_{66}^0; c_{12}^0, c_{13}^0, c_{23}^0. \tag{23}$$

4. Elastic modules and anisotropic behaviour of occasionally heterogeneous media of hexagonal symmetry averaged along axis $y \perp A_6$

First-order system is written as:

$$\frac{d\vec{W}}{dy} = B(y)\vec{W}; \vec{W} = (U_y, \sigma_{yy}, U_x, \sigma_{xy}, U_z, \sigma_{zy})^t. \tag{24}$$

Structure of matrix of coefficients $B(y)$ in (24) is the same as (2). Elements b_{ij} of averaged matrix $B(y)$

$$\langle B \rangle = \frac{1}{h} \int_0^h B(y) dy \tag{25}$$

by inhomogeneity period are written as:

$$\begin{aligned}
\langle b_{12} \rangle &= \left\langle \frac{1}{c_{11}} \right\rangle; b_{13} = ik_x \left\langle \frac{c_{12}}{c_{11}} \right\rangle; \\
b_{15} &= ik_z \left\langle \frac{c_{13}}{c_{11}} \right\rangle; \langle b_{21} \rangle = -\omega^2 \langle \rho \rangle; \\
\langle b_{34} \rangle &= \left\langle \frac{1}{c_{66}} \right\rangle; \langle b_{43} \rangle = -\omega^2 \langle \rho \rangle + k_x^2 \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + k_z^2 \langle c_{44} \rangle; \\
\langle b_{45} \rangle &= k_x k_z \left\langle c_{44} + c_{13} - \frac{c_{12}c_{13}}{c_{11}} \right\rangle; \langle b_{56} \rangle = \left\langle \frac{1}{c_{44}} \right\rangle; \\
\langle b_{65} \rangle &= -\omega^2 \langle \rho \rangle + k_x^2 \langle c_{44} \rangle + k_z^2 \left\langle c_{33} - \frac{c_{13}^2}{c_{11}} \right\rangle.
\end{aligned}$$

Set of equations and vector \vec{w} for anisotropic elastic medium of rhombic symmetry are written as (24).

Structure of B is also written as (2).

Elements b_{ij} for media of rhombic anisotropy:

$$\begin{aligned}
b_{12} &= \frac{1}{c_{22}^0}; b_{13} = ik_x \frac{c_{12}^0}{c_{22}^0}; b_{15} = ik_z \frac{c_{23}^0}{c_{22}^0}; \\
b_{21} &= -\omega^2 \rho_0; b_{24} = ik_x; b_{26} = ik_z; \\
b_{34} &= \frac{1}{c_{66}^0}; b_{43} = -\omega^2 \rho_0 + k_x^2 \left(c_{11}^0 - \frac{c_{12}^0{}^2}{c_{22}^0} \right) + k_z^2 c_{55}^0; \\
b_{45} &= k_x k_z \left(c_{55}^0 + c_{13}^0 - \frac{c_{12}^0 c_{23}^0}{c_{22}^0} \right); b_{56} = \frac{1}{c_{44}^0}; \\
b_{65} &= -\omega^2 \rho_0 + k_x^2 c_{55}^0 + k_z^2 \left(c_{33}^0 - \frac{c_{23}^0{}^2}{c_{22}^0} \right)
\end{aligned} \tag{27}$$

Likewise, comparison of (26) and (27) determine elastic modules of averaged medium.

$$\begin{aligned}
c_{22}^0 &= \left\langle \frac{1}{c_{11}} \right\rangle^{-1}; c_{12}^0 = \left\langle \frac{c_{12}}{c_{11}} \right\rangle \left\langle \frac{1}{c_{11}} \right\rangle^{-1} = \left\langle \frac{c_{12}}{c_{11}} \right\rangle c_{22}^0; \\
c_{13}^0 &= \left\langle \frac{c_{13}}{c_{11}} \right\rangle c_{22}^0; c_{66}^0 = \left\langle \frac{1}{c_{66}} \right\rangle^{-1};
\end{aligned} \tag{28}$$

$$\begin{aligned}
c_{55}^0 &= \langle c_{44} \rangle; c_{44}^0 = \left\langle \frac{1}{c_{44}} \right\rangle^{-1}; c_{11}^0 = \langle c_{11} \rangle - \left\langle \frac{c_{12}^2}{c_{11}} \right\rangle + \frac{c_{12}^0{}^2}{c_{22}^0}; \\
c_{13}^0 &= \langle c_{13} \rangle - \left\langle \frac{c_{12}c_{13}}{c_{11}} \right\rangle + \frac{c_{12}^0 c_{13}^0}{c_{22}^0}; c_{33}^0 = \langle c_{33} \rangle - \left\langle \frac{c_{13}^2}{c_{11}} \right\rangle + \frac{c_{23}^0{}^2}{c_{22}^0}.
\end{aligned}$$

Nine independent elastic modules:

$$c_{11}^0, c_{22}^0, c_{33}^0, c_{44}^0, c_{55}^0, c_{66}^0; c_{12}^0, c_{13}^0, c_{23}^0. \tag{29}$$

Also determine rhombic anisotropy of averaged elastic anisotropic medium of hexagonal symmetry occasionally heterogeneous along axis y .

5. Anisotropic media of rhombic symmetry

Averaging of occasionally heterogeneous anisotropic medium of rhombic symmetry along axis x ends in:

$$\begin{aligned}
\langle B \rangle &= \frac{1}{h} \int_0^h B(x) dx; \\
\langle b_{12} \rangle &= \left\langle \frac{1}{c_{11}} \right\rangle; b_{13} = ik_y \left\langle \frac{c_{12}}{c_{11}} \right\rangle; b_{15} = ik_z \left\langle \frac{c_{13}}{c_{11}} \right\rangle; \\
\langle b_{21} \rangle &= -\omega^2 \langle \rho \rangle; \langle b_{34} \rangle = \left\langle \frac{1}{c_{66}} \right\rangle; \\
\langle b_{43} \rangle &= -\omega^2 \langle \rho \rangle + k_y^2 \langle c_{44} \rangle + \left\langle c_{22} - \frac{c_{12}^2}{c_{11}} \right\rangle k_y^2; \\
\langle b_{56} \rangle &= \left\langle \frac{1}{c_{55}} \right\rangle; \\
\langle b_{45} \rangle &= k_y k_z \left\langle c_{44} + c_{25} - \frac{c_{12}c_{13}}{c_{11}} \right\rangle;
\end{aligned} \tag{30}$$

$$\langle b_{65} \rangle = -\omega^2 \langle \rho \rangle + \langle c_{44} \rangle k_y^2 + \left\langle c_{33} - \frac{c_{13}^2}{c_{11}} \right\rangle k_z^2.$$

Comparison with elements b_{ij} using rhombically anisotropic medium gives:

$$\begin{aligned} \left\langle \frac{1}{c_{11}} \right\rangle &= \frac{1}{c_{11}^0}; \left\langle \frac{c_{12}}{c_{11}} \right\rangle = \frac{c_{12}^0}{c_{11}^0}; \left\langle \frac{c_{13}}{c_{11}} \right\rangle = \frac{c_{13}^0}{c_{11}^0}; \\ \langle \rho \rangle &= \rho_0; \left\langle \frac{1}{c_{66}} \right\rangle = \frac{1}{c_{66}^0}; \langle c_{44} \rangle = c_{44}^0; \\ \left\langle c_{22} - \frac{c_{12}^2}{c_{11}} \right\rangle &= c_{22}^0 - \frac{c_{12}^0{}^2}{c_{11}^0}; \left\langle c_{23} - \frac{c_{12}c_{13}}{c_{11}} \right\rangle = c_{23}^0 - \frac{c_{12}^0 c_{13}^0}{c_{11}^0}; \\ \left\langle \frac{1}{c_{55}} \right\rangle &= \frac{1}{c_{55}^0}; \left\langle c_{33} - \frac{c_{13}^2}{c_{11}} \right\rangle = c_{33}^0 - \frac{c_{13}^0{}^2}{c_{11}^0}. \end{aligned} \quad (31)$$

It follows from these equations:

$$\begin{aligned} c_{11}^0 &= \left\langle \frac{1}{c_{11}} \right\rangle^{-1}; c_{44}^0 = \langle c_{44} \rangle; c_{55}^0 = \left\langle \frac{1}{c_{55}} \right\rangle^{-1}; c_{66}^0 = \left\langle \frac{1}{c_{66}} \right\rangle^{-1}; \\ c_{12}^0 &= \left\langle \frac{c_{12}}{c_{11}} \right\rangle c_{11}^0; c_{13}^0 = \left\langle \frac{c_{13}}{c_{11}} \right\rangle c_{11}^0; c_{22}^0 = \langle c_{22} \rangle - \left\langle \frac{c_{12}^2}{c_{11}} \right\rangle + \frac{c_{12}^0{}^2}{c_{11}^0}; \\ c_{23}^0 &= \langle c_{23} \rangle - \left\langle \frac{c_{12}c_{13}}{c_{11}} \right\rangle + \frac{c_{12}^0 c_{13}^0}{c_{11}^0}; c_{33}^0 = \langle c_{33} \rangle - \left\langle \frac{c_{13}^2}{c_{11}} \right\rangle + \frac{c_{13}^0{}^2}{c_{11}^0}. \end{aligned} \quad (32)$$

Elastic parameters of averaged medium (32) pertain to rhombic anisotropy again. Thus, further averaging of occasionally heterogeneous anisotropic media of rhombic symmetry along one of coordinate will not change anisotropic behaviour of media, only values, the variables of elastic modules change.

In case of averaging along axes y or z this conclusion is correct since anisotropic medium of rhombic symmetry has three orthogonal binary axes along which axes of reference are directed. These axes are equivalent.

6. Conclusion

Given typical scale of inhomogeneity of medium $h \ll \lambda$ of wave length:

- elastic modules of averaging of one-dimensionally heterogeneous isotropic elastic media are determined

- elastic modules of averaging of one-dimensionally heterogeneous media of hexagonal anisotropy are determined:

a. along hexagonal axis

b. along axis orthogonal to hexagonal axis.

- elastic modules of averaging of one-dimensionally heterogeneous media of rhombic anisotropy along binary

axis are determined.

From these results it follows that:

- one-dimensionally heterogeneous isotropic media tend to hexagonal anisotropy;

- bi-dimensionally heterogeneous isotropic media tend to rhombic anisotropy;

- inhomogeneity along hexagonal axis in media of hexagonal anisotropy will not change medium symmetry;

- media of hexagonal anisotropy which are heterogeneous along axes orthogonal to hexagonal axis tend to media of rhombic anisotropy

- inhomogeneity along binary axes in rhombically anisotropic media will not change medium symmetry.

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