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ORIGINAL RESEARCH ARTICLE

Numerical solution of the boundary value problem for the heat equation with fractional Riesz derivative

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ABSTRACT

The work is devoted to the numerical solution of the initial boundary value problem for the heat equation with a fractional Riesz derivative. Explicit and implicit difference schemes are constructed that approximate the boundary value problem for the heat equation with a fractional Riesz derivative with respect to the coordinate. In the case of an explicit difference scheme, a condition is obtained for the time step at which the difference scheme converges. For an implicit difference scheme, a theorem on unconditional convergence is proved. An example of a numerical calculation using an implicit difference scheme is given. It has been established that when passing to a fractional derivative, the process of heat propagation slows down.

Keywords: fractional Riesz derivative; difference scheme; approximation; convergence; thermal transport

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1. Introduction

At the heart of modern science is an experimental dialogue with nature that involves active intervention rather than passive observation. This science sets the task for scientists to learn how to control the physical entity, to force it to act in accordance with the "scenario" that follows from the theoretical scheme put forward by scientists.

In a fractal medium, in contrast to an ordinary continuous medium, a randomly wandering particle moves away from the reference point more slowly, since not all directions of motion become available to it. As shown in Uchaikin and Sibatov[1], the slowdown of the diffusion process in fractal media is so significant that physical quantities begin to change more slowly than in ordinary media, and this effect can be taken into account using integral-differential equations containing a fractional derivative.

Due to the great difficulties that arise in the search for analytical solutions of equations with fractional derivatives, along with analytical methods, numerical methods for solving fractional differential equations are also being developed. The processes of anomalous diffusion and diffusion of particles in inhomogeneous media were numerically studied by Reviznikov and Slastushenskiy[2]. The works of Liu and Xu[3], Meerschaert and Tadjeran[4, 5] as well as our recent paper[6] are also devoted to numerical methods for solving boundary value problems for partial differential equations of fractional order. The paper of Sweis et al.[7] considers a class of differential equations of fractional order with a delay of order ρ and Atangana-Baleanu fractional derivatives in the sense of Caputo fractional derivatives. For the numerical solution, the Galerkin algorithm based on the shifted Legendre polynomials was developed. Convergence and error are proved. Numerical examples are given to determine the efficiency of the algorithm.

In this paper, we construct a numerical solution of the initial-boundary value problem for the heat equation with a fractional Riesz derivative with respect to the spatial coordinate. The fractional Riesz derivative, in contrast to the fractional derivatives of Caputo and Riemann-Liouville, is invariant under the transformation $x \to -x$.

2. Mathematical statement of the problem

Consider in the region $D = \{(x,t) : -L < x < +L, 0 < t \le T\}$ a boundary value problem for the heat equation with a fractional Riesz derivative with respect to the coordinate.

The task. *Find a solution* $u(x,t) \in C^2(D)$ *to the equation:*

$$u_t(x,t) = C(x,t)\,{}^{R}D^{\beta}u(x,t) + f(x,t) \tag{1}$$

where $1 < \beta < 2, C(x,t) \ge 0$ *, satisfying the initial condition* $u(x,0) = \varphi(x)$ *and boundary conditions* $u(-L,t) = \mu_1(t)$ *and* $u(L,t) = \mu_2(t)$*. Here*

$${}^{R}D^{\beta}u(x,t) = \frac{1}{2\Gamma(2-\beta)\cos\left(\frac{(2-\beta)\cdot\pi}{2}\right)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{+\infty} \frac{u(s,t)}{|x-s|^{\beta-1}} ds,$$

is a partial Riesz fractional derivative.

In contrast to the Riemann-Liouville and Caputo fractional order derivatives, the Riesz fractional derivative is invariant under the transformation $x \to -x$, that is, the equality takes place:

$$\frac{\partial^{\beta}u(-x,t)}{\partial(-x)^{\beta}} = \frac{1}{2\Gamma(2-\beta)\cos\left(\frac{(2-\beta)\cdot\pi}{2}\right)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{+\infty} \frac{u(-s,t)}{|x+s|^{\beta-1}} ds =$$

$$= \frac{1}{2\Gamma(2-\beta)\cos\left(\frac{(2-\beta)\cdot\pi}{2}\right)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{+\infty} \frac{u(s,t)}{|x-s|^{\beta-1}} ds = \frac{\partial^{\beta}}{\partial x^{\beta}} u(x,t).$$

Using the expression

$${}^{R}D^{\beta}u(x,t) = \frac{1}{2\cdot\Gamma(2-\beta)\cos\left(\frac{\pi}{2}(2-\beta)\right)} \left({}^{RL}D^{\beta}_{0-}u(x,t) + {}^{RL}D^{\beta}_{0+}u(x,t)\right) \tag{2}$$

Equation (1) will take the form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{C(x,t)}{2\cdot\Gamma(2-\beta)\cos\left(\frac{\pi}{2}(2-\beta)\right)} \left({}^{RL}D^{\beta}_{0+}u(x,t) + {}^{RL}D^{\beta}_{0-}u(x,t)\right) + f(x,t) \tag{3}$$

where

$${}^{RL}D^{\beta}_{0+}u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \frac{u(s,t)}{(x-s)^{\beta-1}} ds, \quad {}^{RL}D^{\beta}_{0-}u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_{x}^{\infty} \frac{u(s,t)}{(x-s)^{\beta-1}} ds.$$

The boundary value problem for Equation (1) will be solved numerically. To do this, we introduce a grid

$$\varpi_{h\tau} = \left\{(x_i, t_n) : x_i = -L + ih, t_n = n\tau, i = 0,1,..., K, h = \frac{2L}{K}, n = 0,1,..., N, \tau = \frac{T}{N}\right\},$$

in the area $\overline{D} = \{(x,t) : -L < x < +L, 0 \le t \le T\}$ with a step h along x and τ along t.

Let $f \in C^2(D)$, then the Grunwald–Letnikov fractional derivatives are defined as

$$u_{a+}^{(\beta)}(x) = \lim_{h \to +0} \frac{1}{h^\beta} \sum_{k=0}^{\left[\frac{x-a}{h}\right]} q_k u(x - kh) \tag{4}$$

$$u_{b-}^{(\beta)}(x) = \lim_{h \to +0} \frac{1}{h^\beta} \sum_{k=0}^{\left[\frac{b-x}{h}\right]} q_k u(x + kh) \tag{4}$$

where $[\bullet]$ is the integer part of number.

According to the Letnikov theorem, if $f \in C^2(D)$ then the Grunwald-Letnikov derivative coincides with the Riemann-Liouville derivative (see Samko et al.[8]).

Using the Grunwald-Letnikov definition of the fractional derivative with respect to space, we replace the fractional derivatives with respect to the coordinate on the right side by the difference expressions:

$$\left(D_+^\beta u\right)_i \sim \frac{1}{h^\beta} \sum_{k=0}^{i+1} q_k u_{i-k+1} = \Lambda_+^\beta u_i \tag{6}$$

$$\left(D_-^\beta u\right)_i \sim \frac{1}{h^\beta} \sum_{k=0}^{K-i+1} q_k u_{i+k-1} = \Lambda_-^\beta u_i \tag{7}$$

For the derivative u_t on the segment $[t_n, t_{n+1}]$ the difference approximation

$$\left(u_t^{/}(x,t)\right)_n \sim \frac{u^{n+1} - u^n}{\tau} \tag{8}$$

$$\frac{u_i^{n+1} - u_i^n}{\tau} =$$

$$= \frac{1}{2 \cdot \Gamma(2-\beta) \cos\left(\frac{\pi}{2}(2-\beta)\right) h^\beta} \left[\sigma \cdot C_i^{n+1}\left(\Lambda_+^\beta u_i^{n+1} + \Lambda_-^\beta u_i^{n+1}\right) + (1-\sigma) \cdot C_i^n \left(\Lambda_+^\beta u_i^n + \Lambda_-^\beta u_i^n\right)\right] + f_i^n,$$

where $C_i^{n+1} \approx C(x_i, t_n)$, $f_i^{n+1} \approx f(x_i, t_n)$, σ is a numeric parameter.

In the case $\sigma = 1$, we obtain an implicit difference scheme with a lead on the template:

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{C_i^{n+1}}{2 \cdot \Gamma(2-\beta) \cos\left(\frac{\pi}{2}(2-\beta)\right) h^\beta} \left[\sum_{k=0}^{i+1} q_k u_{i-k+1}^{n+1} + \sum_{k=0}^{K-i+1} q_k u_{i+k-1}^{n+1}\right] + f_i^n \tag{9}$$

$u_i^0 = \varphi(x_i)$,

$u_0^n = \mu_1(t_n)$,

$u_K^n = \mu_2(t_n)$.

Theorem 1. *Difference Equation (9) is unconditionally stable.*

Proof of Theorem 1. After elementary transformations (9) will take the form:

$$u_i^{n+1} - \xi_i \sum_{k=0}^{i+1} q_k u_{i-k+1}^{n+1} - \xi_i \sum_{k=0}^{K-i+1} q_k u_{i+k-1}^{n+1} = u_i^n + \tau \cdot f_i^{n+1},$$

$u_i^0 = \varphi(x_i)$,

$u_0^n = \mu_1(t_n)$,

$u_K^n = \mu_2(t_n)$,

where $\xi_i = \dfrac{C_i^{n+1}\tau}{2 \cdot \Gamma(2-\beta)\cos\left(\frac{\pi}{2}(2-\beta)\right)h^\beta}$.

The system of equations in matrix form takes the form:

$$AU^{n+1} = U^n + \tau \cdot f^{n+1},$$

where $U^n = \left[u_0^n, u_1^n, ..., u_K^n\right]^T$, $f^n = \left[0, f_1^n, ..., f_{K-1}^n, 0\right]^T$.

The matrix of coefficients looks like:

$$a_{ij} = \begin{cases} 1 - 2\xi_i q_1, & \text{if} \quad j = i \\ -\xi_i (q_2 + q_0), & \text{if} \quad j = i-1 \\ -\xi_i (q_2 + q_0), & \text{if} \quad j = i+1 \\ -\xi_i q_{i-j+1}, & \text{if} \quad j < i-1 \\ -\xi_i q_{j-i+1}, & \text{if} \quad j > i+1 \end{cases},$$

here $a_{00} = 1, a_{0j} = 0$ if $j = 1,2,...,K, a_{KN} = 1$ and $a_{Kj} = 0$ if $j = 0,1,..., K-1, q_1 = -\beta, -q_1 \geq \sum_{k=0,k\neq 1}^{K} q_k$.

Thus, all eigenvalues of the matrix A are in the conjunction K of circles centered at a_{ii} and c with radii:

$$r_i = \frac{1}{\alpha} \sum_{k=0,k\neq i}^{i+1} \xi_i q_k + \sum_{k=0,k\neq i}^{K-i+1} \xi_i q_k < 2\xi_i \beta \tag{10}$$

$$a_{ii} = 1 - 2\xi_i q_1 = 1 - 2\xi_i \beta > 1 \tag{11}$$

It follows from Equations (10) and (11) that the matrix A eigenvalues are greater than 1. Then the matrix A^{-1} eigenvalues are positive and less than 1.

Therefore, difference Equation (9) is unconditionally stable. □

For the case $\sigma = 0$ we get an explicit difference scheme:

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{C_i^n}{2 \cdot \Gamma(2-\beta)\cos\left(\frac{\pi}{2}(2-\beta)\right)h^\beta} \left[\sum_{k=0}^{i+1} q_k u_{i-k+1}^n + \sum_{k=0}^{K-i+1} q_k u_{i+k-1}^n\right] + f_i^n \tag{12}$$

After elementary transformations, the difference Equation (12) will take the form:

$$u_i^{n+1} = (1 + 2q_1\xi_i)u_i^n + \xi_i \left[q_0 (u_{i+1}^n + u_{i-1}^n) + \sum_{k=0}^{i+1} q_k u_{i-k+1}^n + \sum_{k=0}^{K-i+1} q_k u_{i+k-1}^n\right] + \tau \cdot f_i^n \tag{13}$$

Theorem 2. *Difference Equation (13) is stable if* $\dfrac{\tau}{h^\beta} \leq \dfrac{1}{2\beta C_{\max}}$.

Proof of Theorem 2. The finite difference Equation (13) can be reduced to the system of equations:

$$U^{n+1} = AU^n + \tau \cdot f^n.$$

$$U^n=\left[u_0^n,u_1^n,...,u_K^n\right]^T,\ f^n=\left[0,\ f_1^n,...,\ f_{K-1}^n,0\right]^T,\ A=(a_{ij}),\ i,\ j=0,1,...,K.,$$

$$a_{ij}=\begin{cases}0, & \text{if } j\geq i+2\\ 1+2q_1\xi_i, & \text{if } j=i,\\ \xi_i(q_2+q_0), & \text{if } j=i-1\\ \xi_i(q_2+q_0), & \text{if } j=i+1\\ \xi_i q_{i-j+1}, & \text{if } j<i+1\\ \xi_i q_{j-i+1}, & \text{if } j>i+1\end{cases}.$$

Here $a_{00}=1, a_{0j}=0$ if $j=1,2,...,K, a_{KN}=1$ and $a_{Kj}=0$ if $j=0,1,...,K-1, q_1=-\beta$. For $1<\beta\leq 2$ and $i\neq 1$, we have $q_i\geq 0$. Then the eigenvalues of the matrix *A* are found in the conjunction *K* of circles with centers at a_{ii} and radii r_i. According to the well-known theorem about root condition (see, for example, Samarsky and Gulin[9]), the eigenvalues of the matrix *A* are in conjunction *K* with circles centered at a_{ii} and with radii $r_i=\sum_{k=0,k\neq i}^{K} a_{ik}$.

We have

$$a_{ii}=1+2q_1\xi_i=1-2\beta\xi_i \tag{14}$$

$$r_i=\sum_{k=0,k\neq i}^{K} a_{ik}=\xi_i\left(\sum_{k=0,k\neq i}^{i+1} q_k+\sum_{k=0,k\neq i}^{K-i+1} q_k\right)<2\beta\xi_i. \tag{15}$$

It follows from Equations (14) and (15) that in the case $1-4\beta\xi_i>-1$, i.e., $\frac{\tau}{h^\beta}\leq\frac{1}{2\beta C_{\max}}$ all eigenvalues of the matrix *A* are less than 1 in absolute value.

Therefore, the difference method is stable if $\frac{\tau}{h^\beta}\leq\frac{1}{2\beta C_{\max}}$. □

When using an explicit difference scheme to find a numerical solution to the initial-boundary value problem (1), we arbitrarily set a step along the coordinate that satisfies the condition $h<1$. And we find the time step from the condition $\tau\leq\frac{h^\beta}{2\beta C_{\max}}$. For example, let $C_{\max}=1, \beta=1.7, h=0.25$. Then we get $\tau\leq\frac{0.25^{1.7}}{2\cdot 1.7}=0.028$. Therefore, one can take $\tau=0.025$. When implementing the explicit difference scheme algorithm, the number of time grid nodes can be taken equal to $N=\left[\frac{T}{h^\beta/(2\beta C_{\max})}\right]$, where $[\bullet]$ is the integer part of number.

Based on the difference scheme (7), we study the following problem.

$$\frac{\partial T(x,t)}{\partial t}={}^R D^\beta T(x,t)+Q(x) \tag{16}$$

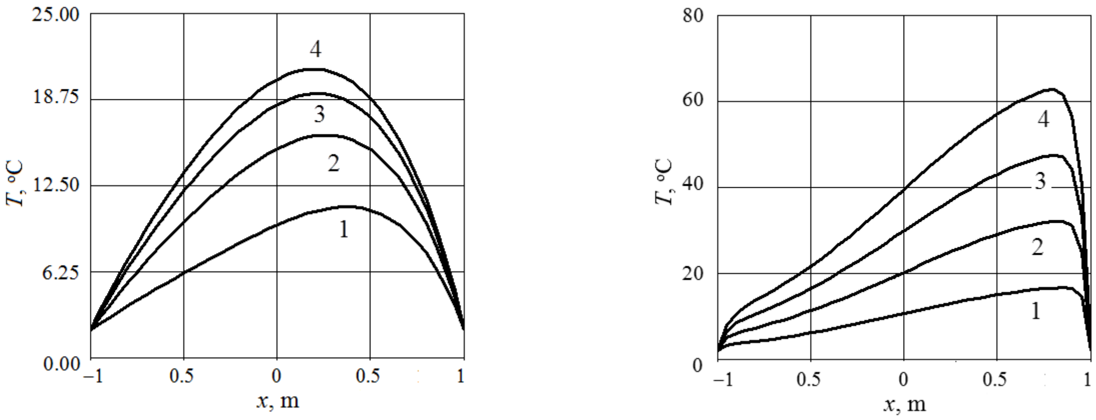
$$T(x,0)=\varphi(x),\ T(-L,t)=\mu_1(t) \text{ and } T(L,t)=\mu_2(t) \tag{17}$$

here $Q(x)=\frac{q}{2\pi\sigma}e^{-\frac{x^2}{2}}$ is a mobile heat source, q is a consumed power, T is a temperature distribution parameter. We consider the case when $L=1$.

3. Results and discussion

Let us consider the case when $q = 100$, $\varphi(x) = 1$, $\mu_1(t) = \mu_2(t) = 2$.

Figure 1 shows the dependences of temperature on the coordinate at different times and for the values of the fractional derivative parameter $\beta = 1.75$ and $\beta = 1.999 \approx 2$. As we see from the figures, the process of temperature distribution slows down when passing to fractional derivatives, which is typical for media with a fractal structure.



(a) **(b)**

Figure 1. Graphs of the dependence of temperature on the coordinate for $\beta = 1.75$ (a) and $\beta = 1.999$ (b) and times: 1 second (1), 2 seconds (2), 3 seconds (3), 4 seconds (4).

4. Conclusions

In this paper, explicit and implicit difference schemes are constructed that approximate the boundary value problem for the heat equation with a fractional Riesz derivative with respect to the coordinate. In the case of an explicit difference scheme, a condition is obtained for a time step under which the difference scheme is stable, and for an implicit difference scheme, an unconditional stability theorem is proved. As an example, we study a boundary value problem for the heat equation with a fractional Riesz derivative with respect to the coordinate and a moving Gaussian heat source. It was found that the process of temperature distribution slows down when passing to fractional derivatives, which is typical for media with a fractal structure. As is known from numerous works, the slowdown of the diffusion process in fractal media is so significant that physical quantities begin to change more slowly than in ordinary media, and this effect can be taken into account only with the help of integral-differential equations containing a fractional derivative.

In future works, we are going to investigate the third initial-boundary value problem for the heat equation with a fractional Riesz derivative with respect to the coordinate. At this stage, an a priori estimate is obtained for solving the differential problem.

Author contributions

Conceptualization, AA and VB; methodology, VB; software, VB; validation, AA and VB; formal analysis, AA and VB; investigation, AA and VB; resources, AA and VB; data curation, AA and VB; writing—original draft preparation, VB; writing—review & editing, AA; visualization, VB; supervision, AA; project administration, AA; funding acquisition, AA and VB. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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