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Unrestricted Pell and Pell-lucas Quaternions

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ABSTRACT

In this study, we define the unrestricted Pell and Pell-Lucas quaternions. We give generating functions, Binet formulas and some generalizations of well-known identities such as Vajda's, Catalan's, Cassini's d'Ocagne's identities.

Keywords: Pell Quaternion; Pell-Lucas Quaternion; Generating Function

1. Introduction

Two well-known integer sequences are Pell and Pell-Lucas sequences. Pell numbers defined by the recurrence relation

$p_n = 2p_{n-1} + p_{n-2}$

where the initial conditions $p_0 = 0$ and $p_1 = 1$. Pell-Lucas numbers satisfy the same recurrence relation, namely

$q_n = 2q_{n-1} + q_{n-2}$

except the initial conditions $q_0 = 1$ and $q_1 = 1$.

Generating functions for the Pell sequence $\{p_n\}$ and the Pell-Lucas sequence $\{q_n\}$ are

$$\sum_{n=0}^{\infty} p_n x^n = \frac{x}{1-2x-x^2} \text{ and } \sum_{n=0}^{\infty} q_n x^n = \frac{2-x}{1-2x-x^2}$$

respectively. Binet formulas for the Pell and Pell-Lucas numbers are

$$p_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } q_n = \frac{\gamma^n + \delta^n}{2}$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. These numbers are roots of the characteristic equation $x^2 - 2x - 1 = 0$. The positive root γ is called "silver ratio" and its role is similar to the golden ratio of Fibonacci numbers. Pell numbers have many interesting properties and we can refer to[4] for details.

Sir W.R. Hamilton introduced the quaternions to extend complex numbers in 1843. The set of quaternions is denoted by

$$\mathrm{H} := \{q : q = a + bi + cj + dk,\ a,b,c,d \in \mathbb{R}\}$$

where i, j, k satisfy

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

Quaternion algebra is a noncommutative division algebra. The conjugate and norm of a quaternion

$q = a + bi + cj + dk$ are

$\bar{q} = a - bi - cj - dk$

and

$N(q) = q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2$

respectively.

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In[3], Horadam gave an idea about Pell and Pell-Lucas quaternions. After him, Cimen and Ipek[2] defined the Pell and Pell-Lucas quaternions as

$Qp_n = p_n + ip_{n+1} + jp_{n+2} + kp_{n+3}$

And

$Qpl_n = pl_n + ipl_{n+1} + jpl_{n+2} + kpl_{n+3}$

where p_n and pl_n are the nth Pell and Pell-Lucas numbers. They also gave a number of properties of these hyper-complex numbers including Binet formulas. Following them, Szynal-Liana and Wloch[5] studied on these type of quaternions. They gave matrix representations and norms of Pell and Pell-Lucas quaternions. Catarino[1] introduced k-Pell and k-Pell-Lucas quaternions and gave some identities for these types of quaternions. Tokeser, Unal and Bilgici[6] defined split Pell and Pell-Luas quaternions over the split quaternion algebra. They gave many identities for these quaternions such as generating functions, Binet formulas, Catalan's, Cassini's, d'Ocagne's identities. In all of these studies, successive Pell and Pell-Lucas numbers were chosen as the coefficients of the standard basis $1, i, j, k$. The contribution of these study is to select these coefficients randomly. The most important coefficients for us will be the real part, namely the coefficient of 1.

2. Preliminaries

In this section, we begin with the definitions of unrestricted Pell and Pell-Lucas quaternions.

Definition 1. For any integers x, y, z and n, the nth unrestricted Pell and Pell-Lucas quaternions are defined by

$$P_n^{(x,y,z)} \coloneqq p_n + ip_{n+x} + jp_{n+y} + kp_{n+z} \quad (1)$$

and

$$Q_n^{(x,y,z)} \coloneqq q_n + iq_{n+x} + jq_{n+y} + kq_{n+z} \quad (2)$$

respectively.

One can easily see that in the case $(x, y, z) = (1,2,3)$ our definitions reduce standard Pell and Pell-Lucas quaternions. By using the identities $p_{-n} = (-1)^{n+1}p_n$ and $q_{-n} = (-1)^n q_n$, we see that

$$P_{-n}^{(x,y,z)} = (-1)^{n+1}\{p_n + (-1)^x p_{n-x} + (-1)^y p_{n-y} + (-1)^z p_{n-z}\}$$

and

$$Q_{-n}^{(x,y,z)} = (-1)^n\{q_n + (-1)^x q_{n-x} + (-1)^y q_{n-y} + (-1)^z q_{n-z}\}.$$

The unrestricted Pell and Pell-Lucas quaternions satisfy the following recurrence relations

$$P_n^{(x,y,z)} = 2P_{n-1}^{(x,y,z)} + P_{n-2}^{(x,y,z)} \text{ and } Q_n^{(x,y,z)} = 2Q_{n-1}^{(x,y,z)} + Q_{n-2}^{(x,y,z)}. \quad (3)$$

Calculating process of generating functions of these types of hyper number sequences is very similar. So, we give the following theorem without proof.

Theorem 2. We have

$$P(x) = (P\_0\text{^}((x,y,z)) + x(P\_1\text{^}((x,y,z)) - 2P\_0\text{^}((x,y,z))))/(1 - 2x - x\text{^}2)$$

and

$$Q(x) = (Q\_0\text{^}((x,y,z)) + x(Q\_1\text{^}((x,y,z)) - 2Q\_0\text{^}((x,y,z))))/(1 - 2x - x\text{^}2)$$

The following theorem gives Binet formulas for the unrestricted Pell and Pell-Lucas quaternions.

Theorem 3. For any integer n, the nth unrestricted Pell and Pell-Lucas quaternions are

$$P_n^{(x,y,z)} = \frac{\hat{\gamma}\gamma^n - \hat{\delta}\delta^n}{\gamma - \delta} \quad (4)$$

and

$$Q_n^{(x,y,z)} = \hat{\gamma}\gamma^n + \hat{\delta}\delta^n \quad (5)$$

where γ and δ are as mentioned above, and

$\hat{\gamma} = 1 + i\gamma^x + j\gamma^y + k\gamma^z$

and

$$\hat{\delta} = 1 + i\delta^x + j\delta^y + k\delta^z.$$

Proof. From the first recurrence relation in Eq.(3), we have

$$\begin{aligned} P_n^{(x,y,z)} &= 2P_{n-1}^{(x,y,z)} + P_{n-2}^{(x,y,z)} \\ &= 2(p_{n-1} + ip_{n-1+x} + jp_{n-1+y} + kp_{n-1+z}) + (p_{n-2} + ip_{n-2+x} + jp_{n-2+y} + kp_{n-2+z}) \\ &= 2p_{n-1} + p_{n-2} + i(2p_{n-1+x} + p_{n-2+x}) + j(2p_{n-1+y} + p_{n-2+y}) + k(2p_{n-1+z} + p_{n-2+z}) \\ &= p_n + ip_{n+x} + jp_{n+y} + kp_n + z \\ &= \frac{1}{\gamma - \delta}[\gamma^n(1 + \gamma^x + \gamma^y + \gamma^z) - \delta^n(1 + \delta^x + \delta^y + \delta^z)]. \end{aligned}$$

The final equation gives Eq.(4). Eq.(5) can be proved similarly.

We need the followings for later use.

Lemma 4. We have

$$\hat{\gamma}\hat{\delta} = \theta + 2\sqrt{2}\eta(6)$$

and

$$\hat{\delta}\hat{\gamma} = \theta - 2\sqrt{2}\eta (7)$$

where

$\theta = -1 - (-1)^x - (-1)^y - (-1)^z + 2Q_0^{(x,y,z)}$

and

$\eta = i(-1)^z p_{y-z} + j(-1)^x p_{z-x} + k(-1)^y p_{x-y}$.

Proof. From the definitions of $\hat{\gamma}$ and $\hat{\delta}$, we have

$$\begin{aligned}
\hat{\gamma}\hat{\delta} &= (1 + i\gamma^x + j\gamma^y + k\gamma^z)(1 + i\delta^x + j\delta^y + k\delta^z) \\
&= 1 - (\gamma\delta)^x - (\gamma\delta)^y - (\gamma\delta)^z \\
&\quad + i(\gamma^x + \delta^x + \gamma^y\delta^z - \gamma^z\delta^y) \\
&\quad + j(\gamma^y + \delta^y + \gamma^z\delta^x - \gamma^x\delta^z) \\
&\quad + k(\gamma^z + \delta^z + \gamma^x\delta^y - \gamma^y\delta^x) \\
&= 1 - (-1)^x - (-1)^y - (-1)^z + iq_x + jq_y + kq_z \\
&\quad + i(\gamma\delta)^z(\gamma^{y-z} - \delta^{y-z}) \\
&\quad + j(\gamma\delta)^x(\gamma^{z-x} - \delta^{z-x}) \\
&\quad + i(\gamma\delta)^y(\gamma^{x-y} - \delta^{x-y}) \\
&= -1 - (-1)^x - (-1)^y - (-1)^z + 2Q_0^{(x,y,z)} \\
&\quad + 2\sqrt{2}\big(i(-1)^z p_{y-z} + j(-1)^x p_{z-x} \\
&\quad + k(-1)^y p_{x-y}\big).
\end{aligned}$$

The last equation gives Eq.(6). Similarly, we can obtain Eq.(7).

From Eqs. (6) and (7), we get the following useful relation

$$\hat{\gamma}\hat{\delta} + \hat{\delta}\hat{\gamma} = 2\theta.$$

3. Results

In this section, we give a number or properties for the unrestricted Pell and Pell-Lucas quaternions. We start with the Vajda's identities.

Theorem 5. (Vajda's İdentity) For any integers n, r, s, x, y and z, we have

$$P_{n+r}^{(x,y,z)} P_{n+s}^{(x,y,z)} - P_n^{(x,y,z)} P_{n+r+s}^{(x,y,z)} = (-1)^n p_r(\theta p_s - 2\eta q_s) \quad (8)$$

and

$$Q_{n+r}^{(x,y,z)} Q_{n+s}^{(x,y,z)} - Q_n^{(x,y,z)} Q_{n+r+s}^{(x,y,z)} = -(-1)^n 2p_r(\theta p_s - 2\eta q_s). \quad (9)$$

Proof. From the Binet formula for the unrestricted Pell quaternions, we have

$$\begin{aligned}
&P_{n+r}^{(x,y,z)} P_{n+s}^{(x,y,z)} - P_n^{(x,y,z)} P_{n+r+s}^{(x,y,z)} \\
&= \frac{1}{8}\big[(\hat{\gamma}\gamma^{n+r} - \hat{\delta}\delta^{n+r})(\hat{\gamma}\gamma^{n+s} - \hat{\delta}\delta^{n+s}) \\
&\quad - (\hat{\gamma}\gamma^n - \hat{\delta}\delta^n)(\hat{\gamma}\gamma^{n+r+s} - \hat{\delta}\delta^{n+r+s})\big] \\
&= \frac{1}{8}\big[-\hat{\gamma}\hat{\delta}\gamma^{n+r}\delta^{n+s} + \hat{\gamma}\hat{\delta}\gamma^n\delta^{n+r+s} - \hat{\delta}\hat{\gamma}\gamma^{n+s}\delta^{n+r} \\
&\quad + \hat{\delta}\hat{\gamma}\gamma^{n+r+s}\delta^n\big] \\
&= \frac{(-1)^n}{8}\big[-\hat{\gamma}\hat{\delta}\gamma^r\delta^s + \hat{\gamma}\hat{\delta}\delta^{r+s} - \hat{\delta}\hat{\gamma}\gamma^s\delta^r \\
&\quad + \hat{\delta}\hat{\gamma}\gamma^{r+s}\big] \qquad (\text{by } \gamma\delta = -1) \\
&= \frac{(-1)^n}{8}\big[-(-1)^r(\hat{\gamma}\hat{\delta}\delta^{s-r} + \hat{\delta}\hat{\gamma}\gamma^{s-r}) + \hat{\gamma}\hat{\delta}\delta^{r+s} \\
&\quad + \hat{\delta}\hat{\gamma}\gamma^{r+s}\big] \\
&= \frac{(-1)^n}{8}\big\{-(-1)^r\big[(\theta + 2\sqrt{2}\eta)\delta^{s-r} \\
&\quad + (\theta - 2\sqrt{2}\eta)\gamma^{s-r}\big] \\
&\quad + (\theta + 2\sqrt{2}\eta)\delta^{r+s} \\
&\quad + (\theta - 2\sqrt{2}\eta)\gamma^{r+s}\big\} \\
&= \frac{(-1)^n}{8}\big[-(-1)^r(2\theta q_{s-r} - 8\eta p_{s-r}) + 2\theta q_{r+s} \\
&\quad - 8\eta p_{r+s}\big] \\
&= \frac{(-1)^n}{8}\big\{2\theta[q_{r+s} - (-1)^r q_{s-r}] \\
&\quad - 8\eta[p_{r+s} - (-1)^r p_{s-r}]\big\}.
\end{aligned}$$

If we substitute the identities $p_{r+s} - (-1)^r p_{s-r} = 2p_r q_s$ and $q_{r+s} - (-1)^r q_{s-r} = 4p_r p_2$ into the last equation, we obtain Eq.(8). Eq.(9) can be obtained in a similar way.

If we take $s = -r$ and use the identities $p_r p_{-r} = -(-1)^r p_r^2$ and $2p_r q_{-r} = (-1)^r p_{2r}$, we obtain the Catalan's identity for the unrestricted Pell and Pell-Lucas quaternions as follows.

Corollary 6. (Catalan's İdentity) For any integers n, r, x, y and z, we have

$$P_{n+r}^{(x,y,z)} P_{n-r}^{(x,y,z)} - \left[P_n^{(x,y,z)}\right]^2 = (-1)^{n+r+1}(\theta p_r^2 + \eta p_{2r}) \quad (10)$$

and

$$Q_{n+r}^{(x,y,z)} Q_{n-r}^{(x,y,z)} - \left[Q_n^{(x,y,z)}\right]^2 = 2(-1)^{n+r}(\theta p_r^2 + \eta p_{2r}). \quad (11)$$

If we take $r = 1$ in Catalan's identity, we obtain Cassini's identities for the unrestricted Pell and Pell-Lucas quaternions.

Corollary 7. (Cassini's Identity) For any integers n, x, y and z, we have

$$P_{n+1}^{(x,y,z)} P_{n-1}^{(x,y,z)} - \left[P_n^{(x,y,z)}\right]^2 = (-1)^n(\theta + 2\eta) \quad (12)$$

and

$$Q_{n+1}^{(x,y,z)}Q_{n-1}^{(x,y,z)} - \left[Q_n^{(x,y,z)}\right]^2 = -2(-1)^n(\theta + 2\eta). \quad (13)$$

Another important identity is d'Ocagne's identity. We give d'Ocagne's identity for the unrestricted Pell and Pell-Lucas quaternions in the following theorem.

Theorem 8. (d'Ocagne's Identity) For any integers m, n, x, y and z, we have

$$P_m^{(x,y,z)}P_{n+1}^{(x,y,z)} - P_{m+1}^{(x,y,z)}P_n^{(x,y,z)} = (-1)^n(\theta p_{m-n} + 2\eta q_{m-n}) \quad (14)$$

and

$$Q_m^{(x,y,z)}Q_{n+1}^{(x,y,z)} - Q_{m+1}^{(x,y,z)}Q_n^{(x,y,z)} = -2(-1)^n(\theta p_{m-n} + 2\eta q_{m-n}). \quad (15)$$

Proof. From the Binet formula for the unrestricted Pell and Pell-Lucas quaternions, we have

$$\begin{aligned} P_m^{(x,y,z)}P_{n+1}^{(x,y,z)} - P_{m+1}^{(x,y,z)}P_n^{(x,y,z)} &= \frac{1}{8}\left[(\hat{\gamma}\gamma^m - \hat{\delta}\delta^m)(\hat{\gamma}\gamma^{n+1} - \hat{\delta}\delta^{n+1}) - (\hat{\gamma}\gamma^{m+1} - \hat{\delta}\delta^{m+1})(\hat{\gamma}\gamma^n - \hat{\delta}\delta^n)\right] \\ &= \frac{1}{8}\left(-\hat{\gamma}\hat{\delta}\gamma^m\delta^{n+1} - \hat{\delta}\hat{\gamma}\gamma^{n+1}\delta^m + \hat{\gamma}\hat{\delta}\gamma^{m+1}\delta^n + \hat{\delta}\hat{\gamma}\gamma^n\delta^{m+1}\right) \\ &= \frac{(-1)^n}{8}\left[\hat{\gamma}\hat{\delta}(\gamma - \delta)\gamma^{m-n} - \hat{\delta}\hat{\gamma}(\gamma - \delta)\delta^{m-n}\right] \\ &= \frac{(-1)^n}{2\sqrt{2}}\left[\hat{\gamma}\hat{\delta}\gamma^{m-n} - \hat{\delta}\hat{\gamma}\delta^{m-n}\right] \\ &= \frac{(-1)^n}{2\sqrt{2}}\left[(\theta + 2\sqrt{2}\eta)\gamma^{m-n} - (\theta - 2\sqrt{2}\eta)\delta^{m-n}\right] \\ &= \frac{(-1)^n}{2\sqrt{2}}\left[\theta(\gamma^{m-n} - \delta^{m-n}) + 2\sqrt{2}\eta(\gamma^{m-n} - \delta^{m-n})\right]. \end{aligned}$$

After some simple elementary operations, we obtain Eq.(14). The other identity can be proved similarly. □

Finally, we give a number of properties of the unrestricted Pell and Pell-Lucas quaternions in the following theorem without proof.

Theorem 9. For any m, n, x, y and z, we have

$$\begin{aligned} &P_{m+k}^{(x,y,z)} = P_m^{(x+k,y+k,z+k)}, \\ &Q_{m+k}^{(x,y,z)} = Q_m^{(x+k,y+k,z+k)}, \\ &P_m^{(x,y,z)} + P_{m-1}^{(x,y,z)} = Q_m^{(x,y,z)}, \\ &Q_m^{(x,y,z)} + Q_{m-1}^{(x,y,z)} = 2P_m^{(x,y,z)}, \\ &P_m^{(x,y,z)} + Q_m^{(x,y,z)} = P_{m+1}^{(x,y,z)}, \\ &P_{m+1}^{(x,y,z)} + P_{m-1}^{(x,y,z)} = 2Q_m^{(x,y,z)}, \\ &Q_{m+1}^{(x,y,z)} + Q_{m-1}^{(x,y,z)} = 4P_m^{(x,y,z)}, \\ &\left[Q_m^{(x,y,z)}\right]^2 - 2\left[P_m^{(x,y,z)}\right]^2 = 2(-1)^m Q_0^{(x,y,z)}, \\ &Q_{m+n}^{(x,y,z)} + (-1)^n Q_{m-n}^{(x,y,z)} = 2q_n Q_m^{(x,y,z)}, \\ &P_{m+n}^{(x,y,z)} + (-1)^n P_{m-n}^{(x,y,z)} = 2q_n P_m^{(x,y,z)}, \\ &P_{m-n}^{(x,y,z)} = (-1)^n(p_{n-1}P_m^{(x,y,z)} - p_n P_{m-1}^{(x,y,z)}). \end{aligned}$$

Conflict of Interest

No conflict of interest was reported by the authors.

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