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H-solvability of Optimal Control Problem for Degenerate Parabolic Variation Inequality

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ABSTRACT

We consider the optimal control problem for degenerate parabolic variation inequality with weight function of potential type that is in differential operator. Using the direct method of calculus of variations we prove the solvability of mentioned above optimal control problem in the class of so-called H-admissible solutions. It is also established that the set of H-admissible pairs is closed in the product of topologies of the state space and the control space.

Keywords: Optimal Control Problem; Parabolic Variation Ineaqulity; Degenerate Weight Function of Potential Type; H-admissible solution; H-optimal Solution

1. Introduction

The main object of investigation is the control problem for degenerate parabolic variation inequality. There are many existence results for evolution variation inequalities without degeneration. But the distinction feature of the considered control object is the fact that its solvability sufficiently depends on properties of some weighted degenerate function. Since this function can be unbounded and reach zero on subsets of some set with zero Lebesgue measure, the operator associated with the problem looses properties which, according to classic theorems, guarantee the solution existence. However, using the so called Hardy-Poincare inequality, one can prove, that the considered optimal control problem for degenerate parabolic variation inequality has a unique optimal solution in weighted Sobolev space [1]. Within the given investigations we propose the alternative approach to the studying of the solvability problem of considered optimal control problem for degenerate parabolic variation inequality. Namely, similarly to[2,3] we introduce the class of so-called H-admissible solutions and by the direct method of calculus of variations we prove the existence of H-optimal solution.

2. Problem definition

Let $\Omega \subset R^N (N \geq 3)$ be bounded open subset with rather smooth boundary $\partial\Omega$ and $0 \in R^N$ be an inner point of Ω Let $Q = (0,T) \times \partial\Omega$ be a cylinder in $R^1 \times R^N$, where $T < +\infty$. As $\Sigma = (0,T) \times \partial\Omega$ we define its lateral surface. Let a function $\rho{:}\Omega \to R$ satisfies the next conditions: $\rho > 0$ a.e. on Ω and

$$\rho \in L^1(\Omega), \rho^{-1} \in L^1(\Omega), \nabla \ln \rho \in L^2(\Omega, R^N),$$
$$\rho + \rho^{-1} \notin L^\infty(\Omega). \quad (1)$$

Thus, the function ρ we can identify with Radon measure Ω on if we consider $\rho(E) = \int_E \rho(x)dx$ for any measurable set $E \subset \Omega$. Let us remind that a non-negative Borel measure which is finite on every compact set we call a non-negative Radon measure on Ω. Henceforth with the function ρwe will connect weight Gilbert spaces $L^2(\Omega, \rho dx)$ and $L^2(\Omega, \rho^{-1}dx)$, where, partialy, $L^2(\Omega, \rho dx)$ be Gilbert space of measurable functions $f : \Omega \to R$, for which

$$\|f\|^2_{L^2(\Omega,\rho dx)} = (f, f)_{L^2(\Omega,\rho dx)} = \int_\Omega f^2 \rho dx < +\infty$$

We also consider the weight Sobolev space $W_0^{1,2}(\Omega, \rho dx)$, which is constituted by such elements from $W_0^{1,1}(\Omega)$, for which the norm

$$\|y\|_{W_0^{1,2}(\Omega,\rho dx)} := \left(\int_\Omega y^2 \rho dx + \int_\Omega |\nabla y|_{R^N}^2 \rho dx \right)^{1/2}$$

is finite, and the weight Sobolev space $H = H(\Omega, \rho\,\mathrm{dx})$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_0^{1,2}(\Omega,\rho dx)}$.

It is known that for any open bounded domain $\Omega \subset R^N$ with rather smooth boundary $\partial\Omega$ there exists the constant $C(\Omega) > 0$ such that the Hardy-Poincare inequality (see [4] for details)

$$\int_\Omega \left[|\nabla y|_{R^N}^2 - \lambda_* \frac{y^2}{|x|_{R^N}^2} \right] dx \ge C(\Omega) \int_\Omega y^2 dx \tag{2}$$

is fulfilled $\forall y \in H_0^1(\Omega),$ where $\lambda_* = (N-2)^2/4$.

Then, as follows from (2) (see [5] for details), $\forall y \in H_0^1(\Omega)$ and $0 < \lambda < \lambda_*$ expressions $\left(\int_\Omega \left[|\nabla y|_{R^N}^2 - \lambda \frac{y^2}{|x|_{R^N}^2} \right] dx \right)^{1/2}$ and $\left(\int_\Omega |\nabla y|_{R^N}^2 dx \right)^{1/2}$ are equivalent norms in Sobolev space $H_0^1(\Omega)$.

Let us consider a nonempty convex closed subset M of the space $L^2(0,T;H)$, which is also sequentially closed with respect to the convergence by the norm:

$$\|y\|_{\rho(0,T)}^2 := \int_0^T \int_\Omega y^2 \rho dxdt + \int_0^T \int_\Omega \left| \nabla y + \frac{y}{2} \nabla \ln \rho \right|_{R^N}^2 \rho dxdt. \tag{3}$$

Let $y_{ad} \in L^2(Q)$, $f \in L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right)$ and $u_0 \in L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right)$ be given distributions, U_∂ be nonempty convex closed subset in $L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right)$ which is defined by the next way:

$$U_\partial = \left\langle u \in L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right) : \|u-u_0\|_{L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right)} \le L \right\rangle, \tag{4}$$

where $\|u\|_{L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right)} = \int_0^T \int_\Omega \frac{u^2}{\rho} dxdt$. Henceforth we consider functions $u \in U_\partial$ as admissible controls.

Let us consider the next optimal control problem for degenerate parabolic inequality with control in right part:

$$I(u,y) = \|y - y_{ad}\|_{L^2\left(0,T;L^2(\Omega,\rho dx)\right)} \to \inf, \tag{5}$$

$$\int_0^T \int_\Omega \dot{v}(v-y) \rho dxdt + \int_0^T \int_\Omega \left(\nabla y, \nabla(v-y)\right)_{R^N} \rho dxdt \ge$$
$$\ge \int_0^T \int_\Omega f(v-y) dxdt + \int_0^T \int_\Omega u(v-y) dxdt, \tag{6}$$

$$v \in M, \dot{v} \in L^2\left(0,\mathrm{T};\left(W_0^{1,2}(\Omega,\rho dx)\right)^*\right), v(0,x) = 0,$$

$$u \in U_\partial,\ y \in M, \tag{7}$$

$$y(0,x) = 0,\ x \in \Omega. \tag{8}$$

Hence, we have the «weak» definition of the optimal control problem for degenerate parabolic inequality (see for details [6]): find such a pair of functions

$$\left(u^0, \mathrm{y}^0\right) \in L^2\left(0,T;L^2\left(\Omega,\rho^{-1}dx\right)\right) \times L^2\left(0,T;\mathrm{W}_0^{1,2}(\Omega,\rho dx)\right),$$

for which relations (6)-(8) are fulfilled and for which the functional (5) reaches its minimal possible value.

3. Analysis of problem (6)-(8)

Similarly to [7, Proposition 1] we can obtain the next result.

Proposition 1. For any element $y \in L^2\left(0,T;\mathrm{W}_0^{1,2}(\Omega,\rho dx)\right)$ the next introduction $y = \dfrac{z}{\sqrt{\rho}}$ takes place and $z = \sqrt{\rho}y \in L^2\left(0,T;\mathrm{W}_0^{1,1}(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$.

Let us note that similarly to [7] we have the next results:

1) there exists such dense set $D_\rho \subset L^2\left(0,T;\mathrm{W}_0^{1,2}(\Omega)\right)$ that $z/\sqrt{\rho} \in L^2\left(0,T;\mathrm{W}_0^{1,2}(\Omega,\rho dx)\right)$, $\forall z \in D_\rho$;

2) we can consider the linear mapping $F : D_\rho \subset L^2\left(0,T;\mathrm{W}_0^{1,2}(\Omega)\right) \to L^2\left(0,T;\mathrm{W}_0^{1,2}(\Omega,\rho dx)\right)$ where $Fz = z/\sqrt{\rho}$ and the conjugated operator

$$F^* : D\left(F^*\right) \subset L^2\left(0,T;\left(\mathrm{W}_0^{1,2}(\Omega,\rho dx)\right)^*\right) \to \quad \to L^2\left(0,T;\left(\mathrm{W}_0^{1,2}(\Omega)\right)^*\right)$$

with respective properties;

3) $\| y \|_{L^2(0,T;W_\rho)} < \infty$, thus $M \subset F\left(D_\rho\right)$, where W_ρ is the closure of the space of finite functions $C_0^\infty(\Omega)$ with respect to the norm

$$\|y\|_\rho^2 := \int_\Omega y^2 \rho dx + \int_\Omega \left|\nabla y + \frac{y}{2}\nabla \ln\rho\right|_{R^N}^2 \rho dx.$$

Definition 1. We say that $\rho : \Omega \to R_+$ be a weight function of potential type, if $\rho > 0$ a.e. on Ω, $\rho \in L^1(\Omega)$, $\rho^{-1} \in L^1(\Omega)$, $\nabla \ln\rho \in L^2\left(\Omega;R^N\right)$ and there exist a constant $\hat{C}(\Omega) > 0$ and a subregion $\Omega_* \subset \Omega$ such that $\rho \in C^1\left(\overline{\Omega \setminus \Omega_*}\right)$, where $dist\left(\partial\Omega,\partial\Omega_*\right) > \delta$ for some $\delta > 0$, and the next relations takes place:

$$\rho(x) \ge \sigma \text{ on } \Omega \setminus \Omega_* \text{ for some } \sigma > 0 \tag{9}$$

$$-\hat{C}(\Omega) \le -\Delta \ln\rho(x) - \frac{1}{2}\left|\nabla \ln\rho\right|_{R^N}^2 < \frac{2\lambda_*}{|x|_{R^N}^2}. \tag{10}$$

The function $V(x) = -\Delta \ln\rho(x) - \dfrac{1}{2}\left|\nabla \ln\rho\right|_{R^N}^2$ is called the Hardy potential for the weight function ρ. Let us represent an equivalent (in some sense) description of the variation inequality (6). Let us construct the set

$$M_1 = \left\{\eta \in L^2\left(0,\mathrm{T};W_0^{1,2}(\Omega)\right) : \eta = \sqrt{\rho}y, \forall y \in M \subset L^2\left(0,\mathrm{T};W_0^{1,2}(\Omega,\rho dx)\right)\right\},$$

which in view of construction and initial assumptions is a convex closed subset of the space $L^2\left(0,\mathrm{T};W_0^{1,2}(\Omega)\right)$. Moreover the element $\eta \in M_1$ "inherits" trace properties of the element $y \in M$ along the boundary $\partial\Omega$.

Let us consider the next problem:

$$\begin{gathered}\int_0^T\!\!\int_\Omega \dot{w}(w-z)dxdt + \int_0^T\!\!\int_\Omega \left(\nabla z,\nabla w - \nabla z\right)_{R^N} dxdt - \\ -\frac{1}{2}\int_0^T\!\!\int_\Omega V(x)z(w-z)dxdt \ge \int_0^T\!\!\int_\Omega \frac{f}{\sqrt{\rho}}(w-z)dxdt + \int_0^T\!\!\int_\Omega p\,(w-z)dxdt, \\ \forall w \in M_1, \dot{w} \in L^2\left(\left(0,\mathrm{T};W_0^{1,2}(\Omega)\right)^*\right), w(0,x) = 0,\end{gathered} \tag{11}$$

$$p \in P_\partial,\ z \in M_1, \tag{12}$$

$$z(0,x) = 0,\ x \in \Omega, \tag{13}$$

where $p = \frac{u}{\sqrt{\rho}}, V(x) = -\Delta \ln \rho - \frac{1}{2}|\nabla \ln \rho|^2_{R^N}$, $P_\partial := \left\{ p \in L^2(Q) : \left\| p - \frac{u_o}{\sqrt{\rho}} \right\|_{L^2(Q)} \le L \right\}.$

In the case if $\rho : \Omega \to R$ is the weight function of potential type the variation inequality (11) has at least one solution for each admissible control $p \in P_\partial$.

Theorem 1. [1, Teorema 1] Let $\rho : \Omega \to R_+$ be a weight function of potential type. Then for given $f \in L^2\left(0,T; L^2\left(\Omega, \rho^{-1}dx\right)\right)$ and $p \in P_\partial$ the variation inequality (11) has a unique solution $z = z(p,f) \in M_1$.

Similarly to [1, Theorem 2], we can obtain the next result about the equivalence (in some sense) for problems (6)-(8) and (11)-(13).

Theorem 2. Let $\rho : \Omega \to R_+$ be a weight function of potential type. Let $f \in L^2\left(0,T; L^2\left(\Omega, \rho^{-1}dx\right)\right)$, $y_{ad} \in L^2(Q)$ be given functions. Then $z \in L^2\left(0,T; \mathrm{W}_0^{1,2}(\Omega)\right)$ is the solution of theproblem (11)-(13) for given $p \in P_\partial$ if and only if $y \in L^2\left(0,T; \mathrm{W}_0^{1,2}(\Omega, \rho dx)\right)$ is the solution of the problem (6)-(8) for given $u \in U_\partial$.

4. Topological property of the set of H-admissible solutions

Definition 2. We say that the set

$$\Xi_H = \left\{ (\mathrm{u},\mathrm{y}) \in L^2\left(0,T; L^2\left(\Omega, \rho^{-1}dx\right)\right) \times L^2(0,T;H) \,|\, (\mathrm{u},\mathrm{y}) \text{ is concerned by } (6)-(8) \right\}$$

is called the set of H-admissible solutions.

Definition 3. The pair $(\mathrm{u}^0, \mathrm{y}^0) \in L^2\left(0,T; L^2\left(\Omega, \rho^{-1}dx\right)\right) \times L^2(0,T;H)$ is an H-optimal solution of the problem (5)-(8), if $(\mathrm{u}^0, \mathrm{y}^0) \in \Xi_H$ and $I(\mathrm{u}^0, \mathrm{y}^0) = \inf_{(\mathrm{u},\mathrm{y}) \in \Xi_H} I(\mathrm{u},\mathrm{y})$.

Definition 4. We say that the sequence $\{y_k\}_{k \ge 1} \subset L^2(0,T;H)$ is weakly convergent to an element $y \in L^2(0,T;H)$ as $k \to \infty$, if the given sequence is bounded and $y_k \to y$ weakly in $L^2(0,T; L^2(\Omega, \rho dx))$ and $\nabla y_k \to \nabla y$ weakly in $L^2(0,T; L^2(\Omega, \rho dx)^N)$, $k \to \infty$.

Remark 1. Let us consider the space X^2_ρ that is the closure of the set $K = \left\{ (\mathrm{y}, \nabla \mathrm{y}),\ \mathrm{y} \in L^2(0,T; C_0^\infty(\Omega)) \right\}$ in $L^2(0,T;H) \times L^2\left(0,T; L^2(\Omega, \rho dx)^N\right)$. Similarly to [8, 9] we can assume that the space X^2_ρ is closed in $L^2(0,T; L^2(\Omega, \rho dx)) \times L^2(0,T; L^2(\Omega, \rho dx)^N)$.

Let τ be the topology on $L^2(0,T; L^2(\Omega, \rho^{-1}dx)) \times L^2(0,T;H)$ which we define as the product of the weak topology of the space $L^2(0,T; L^2(\Omega, \rho^{-1}dx))$ and the weak topology of $L^2(0,T;H)$.

Theorem 3. Let $\rho(\mathrm{x}) > 0$ be a degenerate weight function of potential type. Then for every $f \in L^2(0,T; L^2(\Omega, \rho^{-1}\,\mathrm{dx}))$ the set Ξ_H is sequentially τ-closed.

Proof. Let $\{(\mathrm{u}_k, \mathrm{y}_k)\}_{k \ge 1} \subset \Xi_H$ be any τ-convergent sequence of admissible pairs of the problem (5)-(8) (in view of Theorems 1, 2 such choice is always possible). Let $\{(\mathrm{u}_0, \mathrm{y}_0)\}$ be its τ-limit. Let us show that $\{(\mathrm{u}_0, \mathrm{y}_0)\} \in \Xi_H$.

Since M and U_∂ are convex closed sets in $L^2(0,T;H)$ and $L^2(0,T;L^2(\Omega,\rho^{-1}\,dx))$ respectively, then by Mazur lemma they are weakly closed. Thus, $u_0 \in U_\partial$, $y_0 \in M$. Let us show that the limit pair satisfies (6). Since $\{(u_k, y_k)\}_{k\geq 1}$ is admissible pair for the problem (5)-(8), then

$$\int_0^T\int_\Omega \dot{v}(v-y_k)\rho dxdt + \int_0^T\int_\Omega (\nabla y_k, \nabla(v-y_k))_{\mathbb{R}^N}\rho dxdt \geq \int_0^T\int_\Omega f(v-y_k)dxdt + \int_0^T\int_\Omega u_k(v-y_k)dxdt. \quad (14)$$

Let us consider the right part of (14), rewriting it in the next way:

$$\int_0^T\int_\Omega f\, y_k\, dxdt - \int_0^T\int_\Omega fvdxdt + \int_0^T\int_\Omega u_k\, y_k\, dxdt - \int_0^T\int_\Omega u_k vdxdt = I_1 + I_2 + I_3 + I_4.$$

Let us consider I_3:

$$I_3 = \int_0^T\int_\Omega u_k\, y_k\, dxdt \pm \int_0^T\int_\Omega u_k\, y_0\, dxdt = \int_0^T\int_\Omega u_k(y_k - y_0)dxdt + \int_0^T\int_\Omega u_k\, y_0\, dxdt.$$

In the case when $\rho^{-1}\in L^1(\Omega)$ the space H is continuously embedded into $W_0^{1,1}(\Omega)$ (see [2] for details) , the space $W_0^{1,1}(\Omega)$ is compactly embedded into $L^1(\Omega)$. Thus, we have $y_k \to y_0$ in $L^1(\Omega)$ and up to the subsequence $y_k \to y_0$ a.e. in Ω. Hence, $\int_0^T\int_\Omega u_k(y_k - y_0)dxdt \to 0$, $k\to\infty$. In view of τ-convergence of $\{(u_k, y_k)\}_{k\geq 1}$ and that fact that $L^2(0,T;L^2(\Omega,\rho^{-1}dx))$ is the conjugate space to $L^2(0,T;L^2(\Omega,\rho dx))$ (see [10] for details), we have that $I_1 \to \int_0^T\int_\Omega fy_0 dxdt$, $I_4 \to -\int_0^T\int_\Omega u_0 vdxdt$, $\int_0^T\int_\Omega u_k y_0 dxdt \to \int_0^T\int_\Omega u_0 y_0 dxdt$ as $k\to\infty$. Thus,

$$\lim_{k\to\infty}\int_0^T\int_\Omega (f+u_k)(y_k - v)dxdt = \int_0^T\int_\Omega (f+u_0)(y_0 - v)dxdt. \quad (15)$$

Now we pass to the limit in the relation (14) as $k\to\infty$, using (15) and the property of lower semicontinuity in $L^2(0,T;L^2(\Omega,\rho\,dx)^N)$ with respect to the weak convergence (see [3] for details). As a result we have

$$\int_0^T\int_\Omega \dot{v}(y_0 - v)\rho\, dxdt + \int_0^T\int_\Omega (\nabla y_0, \nabla(y_0 - v))_{\mathbb{R}^N}\rho\, dxdt = \int_0^T\int_\Omega \dot{v}y_0\rho\, dxdt + \int_0^T\int_\Omega \dot{v}\, v\,\rho\, dxdt +$$

$$+\int_0^T\int_\Omega (\nabla y_0, \nabla y_0)_{\mathbb{R}^N}\rho\, dxdt - \int_0^T\int_\Omega (\nabla y_0, \nabla v)_{\mathbb{R}^N}\rho\, dxdt \leq$$

$$\leq \lim_{k\to\infty}\int_0^T\int_\Omega \dot{v}y_k\rho dxdt + \lim_{k\to\infty}\int_0^T\int_\Omega \dot{v}v\rho dxdt + \varliminf_{k\to\infty}\int_0^T\int_\Omega (\nabla y_k, \nabla y_k)_{\mathbb{R}^N}\rho dxdt - \lim_{k\to\infty}\int_0^T\int_\Omega (\nabla y_k, \nabla v)_{\mathbb{R}^N}\rho dxdt \leq$$

$$\leq \varliminf_{k\to\infty}\left(\int_0^T\int_\Omega \dot{v}y_k\rho dxdt + \int_0^T\int_\Omega \dot{v}v\rho dxdt + \int_0^T\int_\Omega (\nabla y_k, \nabla y_k)_{\mathbb{R}^N}\rho dxdt - \int_0^T\int_\Omega (\nabla y_k, \nabla v)_{\mathbb{R}^N}\rho dxdt\right) \leq$$

$$\leq \varliminf_{k\to\infty}\left(\int_0^T\int_\Omega (f+u_k)(y_k - v)dxdt\right) = \lim_{k\to\infty}\left(\int_0^T\int_\Omega (f+u_k)(y_k - v)dxdt\right) = \int_0^T\int_\Omega (f+u_0)(y_0 - v)dxdt.$$

Hence, the τ-limit pair (u_0, y_0) is H-admissible for the problem (5)-(8), and $(u_0, y_0)\in \Xi_H$.

The theorem is proved.

5. H-solvability of the problem (5)-(8)

Theorem 4. Let $\rho(x) > 0$ be a degenerate weight function of potential type. Then for every $f\in L^2(0,T;L^2(\Omega,\rho^{-1}\,dx))$ the set of H-optimal solutions is nonempty.

Proof. Let us note that the cost functional (5) is lower τ -semicontinuous on Ξ_H. Let $\{u_k, y_k\}_{k\geq 1} \subset \Xi_H$ - H-minimizing sequence of the problem (5)-(8), that is

$$\lim_{k\to\infty} I(u_k, y_k) = \inf_{(u,y)\in\Xi_H} I(u,y) < +\infty.$$

From the definition of the set U_∂ we have that the sequence $\{u_k\}_{k\geq 1}$ is bounded in $L^2(0,T;L^2(\Omega,\rho^{-1}dx))$. Thus up to a subsequence there exists an element $u^* \in U_\partial$ such that $u_k \to u^*$ weakly in $L^2(0,T;L^2(\Omega,\rho^{-1}dx))$ as $k\to\infty$. Let us prove the boundedness of the sequence $\{y_k = y(u_k)\}_{k\geq 1}$ in the space $L^2(0,T;H)$. From upper assumptions we have that $y \in M$ can be represented as $y = z/\sqrt{\rho}$, where $z \in L^2(0,T;H_0^1(\Omega))$ and

$$\|z\|_{L^2(0,T;H_0^1(\Omega))} = \int_0^T\int_\Omega y^2 \rho dxdt + \int_0^T\int_\Omega \left|\nabla(\sqrt{\rho}\, y)\right|^2_{\mathbb{R}^N} dxdt. \tag{16}$$

Moreover, it is known that the sequence $\{z_k\}_{k\geq 1}$ is bounded in the space $L^2(0,T;H_0^1(\Omega))$ [see, for example the proof of Theorem from [7] for elliptic case]. In view of (16) we have, particularly, the boundedness of the sequence $\{y_k\}_{k\geq 1}$ in the space $L^2(0,T;L^2(\Omega;\rho dx))$.

Let us prove the boundedness of the sequence $\{\nabla y_k\}_{k\geq 1}$ in the space $L^2(0,T;L^2(\Omega;\rho dx)^N)$. Using the equivalent representation of the problem (6)-(8) properties of the operator $\Lambda = d/dt$ (see [1] for details), Proposition 1, boundedness of the sequence $\{z_k\}_{k\geq 1}$ in the space $L^2(0,T;H_0^1(\Omega))$ and inequality

$$\int_0^T\int_\Omega (\nabla y, \nabla(y - v_0))_{\mathbb{R}^N}\, \rho\, dxdt \geq \|\nabla y\|^2_{L^2(0,T;L^2(\Omega;\rho dx)^N)} - \|\nabla y\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)} \|\nabla v_0\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)},$$

we obtain the next property:

$$\frac{\int_0^T\int_\Omega \dot{v}_0 (y - v_0)\rho\, dxdt + \int_0^T\int_\Omega (\nabla y, \nabla(y - v_0))_{\mathbb{R}^N}\rho\, dxdt}{\|\nabla y\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}} \to +\infty, \tag{17}$$

as $\|\nabla y\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)} \to \infty$ for some $v_0 \in M$ such that, $\dot{v}_0 \in L^2(0,T;H^*)$.

Let us suppose that there exists a subsequence $\{\nabla y_{k_n}\}_{n\geq 1} \subset \{\nabla y_k\}_{k\geq 1}$ such that $\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)} \to \infty$ as $n\to\infty$, and, using (17) we obtain:

$$+\infty \leftarrow \frac{\int_0^T\int_\Omega \dot{v}_0 (y_{k_n} - v_0)\rho\, dxdt + \int_0^T\int_\Omega (\nabla y_{k_n}, \nabla(y_{k_n} - v_0))_{\mathbb{R}^N}\rho\, dxdt}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}} \leq \frac{\int_0^T\int_\Omega (f + u_{k_n})(y_{k_n} - v_0)dxdt}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}} \leq$$

$$\leq \frac{\|f + u_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))} \|y_{k_n} - v_0\|_{L^2(0,T;L^2(\Omega;\rho dx))}}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}} \leq$$

$$\leq \frac{\left(\|f\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))} + \|u_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))}\right)\left(\|y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx))} + \|v_0\|_{L^2(0,T;L^2(\Omega;\rho dx))}\right)}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}} \leq$$

$$\leq \frac{\left(\|f\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))}+\|u_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))}\right)\left(\|y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx))}+\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}+\|v_0\|_{L^2(0,T;L^2(\Omega;\rho dx))}\right)}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}}=$$

$$=\left(\|f\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))}+\|u_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho^{-1}dx))}\right)\left(\frac{\|y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx))}}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}}+1+\frac{\|v_0\|_{L^2(0,T;L^2(\Omega;\rho dx))}}{\|\nabla y_{k_n}\|_{L^2(0,T;L^2(\Omega;\rho dx)^N)}}\right)\leq \overline{C}$$

for an arbitrary fixed element $v_0 \in L^2(0,T;H)$, since the set U_∂ is bounded in the space $L^2(0,T;L^2(\Omega;\rho^{-1}dx))$. We have the contradiction. Thus, we obtained the boundedness of the sequence $\{\nabla y_{k_n}\}_{n\geq 1}$ in the space $L^2(0,T;L^2(\Omega;\rho\, dx)^N)$. Hence, up to a subsequence there exists an element $y^* \in L^2(0,T;H)$ such that $y_k \to y^*$ weakly in $L^2(0,T;L^2(\Omega;\rho\, dx))$ and $\nabla y_k \to \nabla y^*$ weakly in $L^2(0,T;L^2(\Omega;\rho\, dx)^N)$ (see Remark 1) as $k\to\infty$. In view of Theorem 3 the set Ξ_H is sequentially τ-closed then the pair (u^*,y^*) is H-admissible for the problem (6)-(8). From the lower τ-semicontinuity of the cost functional (5) we have that $I(u^*,y^*)\leq \underline{\lim}_{k\to\infty} I(u_k,y_k)=\inf_{(u,y)\in\Xi_H} I(u,y)$. Hence, (u^*,y^*) is the H-optimal pair.

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