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An Appropriate Numerical Method for Solving Nonlinear Volterra-Fredholm Integral Equations

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ABSTRACT

This paper is concerned with the numerical solution of the mixed Volterra-Fredholm integral equations by using a version of the block by block method. This method efficient for linear and nonlinear equations and it avoids the need for spacial starting values. The convergence is proved and finally performance of the method is illustrated by means of some significative examples.

***Keywords*:** Volterra-Fredholm integral equations; Romberg quadrature rule; numerical solution.

1. Introduction

In this paper we illustrate a numerical method for linear and nonlinear Volterra-Fredholm integral equations of the form

$$u(t,x) = f(t,x) + \int_0^t \int_\Omega k(t,s,x,\xi) d\xi ds, \qquad t \in I := [0,T], x \in \Omega \quad (1.1)$$

where Ω is a closed subset on $R^d \ (d=1,2)$. For linear case we can write

$$k(t,s,x,\xi,u(s,\xi)) = H(t,s,x,\xi)u(s,\xi).$$

It is assumed in the following that the given real-valued functions $f(t,x)$ and $k(t,s,x,\xi,u(s,\xi))$ are at least continuous on $D := [0,T] \times \Omega$ and $S \times R$ where $S := \{(t,s,x,\xi) : 0 \le s \le t \le T; (x,\xi) \in \Omega \times \Omega\}$ and are such that (1.1) possesses a unique solution $u \in C(D)$. Existence and uniqueness results for linear and nonlinear cases may be found in[5, 8, 10, 13].

An integral equation of mixed Volterra-Fredholm type describe the spatio-temporal development of an epidemic[14] and various physical and biological problems[5, 14].

The literature on numerical methods for solving these type of equations are briefly described as follows:

A continuous time collocate method is investigated in[10, 1] for linear and nonlinear cases. In[7] a particular trapezoidal Nystrom method are considered with their asymptotic error expansions. In[15, 17], Adomian decomposition method and Homotopy perturbation method is applied to obtain numerical solution. Also methods based on the Taylor polynomials, radial basis functions and Legendre wavelets respectively proposed in[16, 4, 18]. Recently in[12, 6] authors using Tau-collocation method and hybrid Legendre for solving these equations. In the following, we are going to introduce a block by block method for solving these mixed equations. This method produces a block of values at a time and it has the following extra advantages:

1). Simplicity of application.

2). Don't need to special starting values.

3). Fixed dimension of the resultant system in each step. The dimension of these systems depends on the spatial mesh and the blocks number of the method. Of course, for the very large spatial meshes, we can use the iterative method for solving resultant system[3].

4). Increase the order of convergence by increasing number of blocks.

The rest of the paper is organized as follows. In Section 2, we describe a numerical method, afterward we prove the convergence result in Section 3 and finally, a comparison between the results of the Adomian decomposition method and the proposed method reported in Section 4.

2. Description of the method

We shall restrict our description to the cases where Ω is one dimensional spatial domain $\Omega=[a,b]\subset ¡$. The extension to the spatial domain $\Omega\subset ¡^{2}$ dose not present essential difficulties. Suppose that the domain Ω is divided up by an uniform grid $\Omega_h:=\{x_0=a,x_1,...,x_{4N}=b\}$ with step size h. The number of mesh points is multiple of 4, since we put the number of blocks equal to 4 (for 8, 16 ... blocks, the process is similar). Collocate (1.1) relative to spatial variable (x), then

$$
\begin{aligned}
u(t,x_i) &= f(t,x_i)+\int_0^t\int_{x_0}^{x_{4N}} k(t,s,x_i,\xi,u(s,\xi))d\xi ds \\
&= f(t,x_i)+\int_0^t\int_{x_0}^{x_4} k(t,s,x_i,\xi,u(s,\xi))d\xi+\int_0^t\int_{x_4}^{x_8} k(t,s,x_i,\xi,u(s,\xi))d\xi ds+... \\
&+\int_0^t\int_{x_{4N-4}}^{x_{4N}} k(t,s,x_i,\xi,u(s,\xi))d\xi ds \qquad i=0,1,...,4N. \qquad (2.1)
\end{aligned}
$$

Approximating inner integrals by two-step Romberg quadrature rule[9] lead to

$$
\begin{aligned}
u_i(t) &= f(t,x_i)+\frac{14}{45}h\int_0^t k(t,s,x_i,x_0,u_0(s))ds-\frac{14}{45}h\int_0^t k(t,s,x_i,x_{4N},u_{4N}(s))ds \\
&+\frac{28}{45}h\sum_{j=1}^{N}\int_0^t k(t,s,x_i,x_{4j},u_{4j}(s))ds+\frac{24}{45}h\sum_{j=1}^{N}\int_0^t k(t,s,x_i,x_{4j-2},u_{4j-2}(s))ds \\
&+\frac{64}{45}h\sum_{j=1}^{N}\int_0^t k(t,s,x_i,x_{4j-1},u_{4j-1}(s))ds+\frac{64}{45}h\sum_{j=1}^{N}\int_0^t k(t,s,x_i,x_{4j-3},u_{4j-3}(s))ds, \\
&\qquad i=0,1,...,4N, \qquad (2.2)
\end{aligned}
$$

where $u_i(t)\;;\;u(t,x_i)$.

Let $\Pi_M:=\{t_m:0=t_0<t_1<...<t_{4M}=T\}$ be a partition of the time interval [0,T] with constant step size $\tau=t_{m+1}-t_m,\ m=0,1,...,4M$ and set t=t_{4m+p} (m=0,1,...,4M-1; p=1,2,3,4), then

$$
\begin{aligned}
u_i(t_{4m+p}) = \quad & f(t_{4m+p},x_i)+\frac{14}{45}h\int_0^{t_{4m+p}} k(t_{4m+p},s,x_i,x_0,u_0(s))ds \\
- \quad & \frac{14}{45}h\int_0^{t_{4m+p}} k(t_{4m+p},s,x_i,x_{4N},u_{4N}(s))ds+\frac{28}{45}h\sum_{j=1}^{N}\int_0^{t_{4m+p}} k(t_{4m+p},s,x_i,x_{4j},u_{4j}(s))ds \\
+ \quad & \frac{24}{45}h\sum_{j=1}^{N}\int_0^{t_{4m+p}} k(t_{4m+p},s,x_i,x_{4j-2},u_{4j-2}(s))ds \\
+ \quad & \frac{64}{45}h\sum_{j=1}^{N}\int_0^{t_{4m+p}} k(t_{4m+p},s,x_i,x_{4j-1},u_{4j-1}(s))ds \\
+ \quad & \frac{64}{45}h\sum_{j=1}^{N}\int_0^{t_{4m+p}} k(t_{4m+p},s,x_i,x_{4j-3},u_{4j-3}(s))ds, \qquad i=0,1,...,4N. \qquad (2.3)
\end{aligned}
$$

By separating the integrals on intervals $[0,t_{4m}]$ and $[t_{4m},t_{4m+p}]$ we will have

$$
\begin{aligned}
u_i(t_{4m+p}) = \quad & f(t_{4m+p}, x_i) + \frac{14}{45} h \int_0^{t_{4m}} k(t_{4m+p}, s, x_i, x_0, u_0(s)) ds \\
& + \frac{14}{45} h \int_{t_{4m}}^{t_{4m+p}} k(t_{4m+p}, s, x_i, x_0, u_0(s)) ds + ... \\
& + \frac{64}{45} h \sum_{j=1}^{N} \int_0^{t_{4m}} k(t_{4m+p}, s, x_i, x_{4j-3}, u_{4j-3}(s)) ds \\
& + \frac{64}{45} h \sum_{j=1}^{N} \int_{t_{4m}}^{t_{4m+p}} k(t_{4m+p}, s, x_i, x_{4j-3}, u_{4j-3}(s)) ds. \qquad (2.4)
\end{aligned}
$$

using two-step Romberg quadrature rule to approximate integrals on interval $[0, t_{4m}]$ yields

$$
\begin{aligned}
& \frac{14}{45} h \int_0^{t_{4m}} \left[k(t_{4m+p}, s, x_i, x_0, u_0(s)) - k(t_{4m+p}, s, x_i, x_{4N}, u_{4N}(s)) \right] ds \\
& + \frac{h}{45} \sum_{j=1}^{N} \int_0^{t_{4m}} \Big[28k(t_{4m+p}, s, x_i, x_{4j}, u_{4j}(s)) + 24k(t_{4m+p}, s, x_i, x_{4j-2}, u_{4j-2}(s)) \\
& +64(k(t_{4m+p}, s, x_i, x_{4j-1}, u_{4j-1}(s)) + k(t_{4m+p}, s, x_i, x_{4j-3}, u_{4j-3}(s))) \Big] ds \\
& ; \; \frac{14}{4050} h t_{4m} \sum_{l=0}^{4} v_l \left[k(t_{4m+p}, t_{lm}, x_i, x_0, u_{lm,0}) - k(t_{4m+p}, t_{lm}, x_i, x_{4N}, u_{lm,4N}) \right] \\
& + \frac{h t_{4m}}{4050} \sum_{j=1}^{N} \sum_{l=0}^{4} \sum_{q=0}^{3} v_l w_q k(t_{4m+p}, t_{lm}, x_i, x_{4j-q}, u_{lm,4j-q}), \\
& \qquad\qquad m = 0,1,...,M-1, \; p = 0,1,...,4, \qquad (2.5)
\end{aligned}
$$

where $v_0 = v_4 = 7, \; v_1 = v_3 = 32, v_2 = 12, w_0 = 28, w_1 = w_3 = 64, w_2 = 24$ and $u_{i,j} \; ; \; u(t_i, x_j)$. Similarly, for the other interval $([\mathrm{t}_{4m}, \mathrm{t}_{4m+p}])$ we can write

$$
\begin{aligned}
& \frac{14}{45} h \int_{t_{4m}}^{t_{4m+p}} \left[k(t_{4m+p}, s, x_i, x_0, u_0(s)) - k(t_{4m+p}, s, x_i, x_{4N}, u_{4N}(s)) \right] ds \\
& + \frac{h}{45} \sum_{j=1}^{N} \int_{t_{4m}}^{t_{4m+p}} \Big[28k(t_{4m+p}, s, x_i, x_{4j}, u_{4j}(s)) + 24k(t_{4m+p}, s, x_i, x_{4j-2}, u_{4j-2}(s)) \\
& +64(k(t_{4m+p}, s, x_i, x_{4j-1}, u_{4j-1}(s)) + k(t_{4m+p}, s, x_i, x_{4j-3}, u_{4j-3}(s))) \Big] ds \\
& ; \; \frac{14}{4050} h t_p \sum_{l=0}^{4} v_l \left[k(t_{4m+p}, t_{4m+lp/4}, x_i, x_0, u_{4m+lp/4,0}) - k(t_{4m+p}, t_{4m+lp/4}, x_i, x_{4N}, u_{4m+lp/4,4N}) \right] \\
& + \frac{h t_p}{4050} \sum_{j=1}^{N} \sum_{l=0}^{4} \sum_{q=0}^{3} v_l w_q k(t_{4m+p}, t_{4m+lp/4}, x_i, x_{4j-q}, u_{4m+lp/4,4j-q}), \\
& \qquad\qquad m = 0,1,...,M-1, \; p = 0,1,...,4. \qquad (2.6)
\end{aligned}
$$

If $lp/4 (l, p = 1,2,3)$ not to be integer, then $t_{4m+lp/4} \notin \Pi_M$ and $u_{4m+lp/4,:}$ will be unknown which leads to a difficulty in computing (2.6). In these cases, we will use Lagrange interpolation at the points t_{4m}, t_{4m+1}, t_{4m+2}, t_{4m+3} and t_{4m+4}; that is

$$u_{4m+lp/4,:} \; ; \; P(t_{4m+lp/4},:) = \sum_{j=0}^{4} L_j(lp/4) u_{4m+j,:}$$

Where

$$L_j(lp/4) = \prod_{\substack{i=0 \\ i \neq j}}^{4} \frac{lp/4 - i}{j - i}, \qquad j = 0,1,\ldots,4.$$

Finally, by substituting (2.5) and (2.6), in relation (2.4) we will have a system of 4(4N+1) equations which is solved for $u_{4m+1,i}, u_{4m+2,i}, u_{4m+3,i}$ and $u_{4m+4,i}$ $(i = 0,1,\ldots,4N)$ in each step. The dimension of the resultant system depends on the spatial mesh and for large amounts of N we will have a large system in each step. The solution of these systems can hence be costly, for reduce these costs we can use of the fast iterative method for large systems[3].

3. Convergence analysis

Theorem 3.1 Assume that P is the order of the Nystrom method (2.2) and let $u_{n,m}$ be the discrete time solution to (2.2) in (t_n, x_m) obtained by a DQ (Direct Quadrature) method of order Q. Then for $u \in C^{P,Q}([0,T]\times\Omega),$ the total error E(t,x) satisfies $|E(t,x)| = O(h^P) + O(\tau^Q), (t,x) \in \Pi_M \times \Omega_h.$

Proof: [2].

Remark 3.2 We used two-step Romberg quadrature rule with 6 order of convergence for obtain (2.2). Also, the order of convergence of the block by block method is $O(\tau^6)$ [9]. Then for $u \in C^{6,6}([0,T]\times\Omega),$ the total error satisfies

$$|E(t,x)| = O(h^6) + O(\tau^6), \qquad (t,x) \in \Pi_M \times \Omega_h.$$

4. Numerical illustration

In this section we present performance of the proposed solution. All calculations is supported by using Maple 14 and the test problems is selected from the literature.

Example 1. Consider the nonlinear Volterra-Fredholm integral equation

$$u(t,x) = f(t,x) + \int_0^t \int_\Omega H(t,s,x,\xi)(1 - exp(-u(s,\xi)))d\xi ds, \qquad (t,x) \in \Omega\times[0,1],$$

with $\Omega = [0,1]$,

$$H(t,s,x,\xi) = \frac{x(1-\xi^2)}{(1+t)(1+s^2)},$$

$$f(t,x) = -log(1 + \frac{xt}{1+t^2}) + \frac{xt^2}{8(1+t)(1+t^2)}.$$

The exact solution for this equation is $u(t,x) = -log(1 + xt/(1+t^2))$ [1].

A comparison between the absolute errors of the Adomian decomposition method (reported in [11]) and the suggested method are shown in Table 1. The Adomian's method applied with two iterations and for the block by block method we chose M=N=2.

Example 2. Consider the linear Volterra-Fredholm integral equation ([2]),

$$u(t,x) = f(t,x) + \int_0^t \int_0^2 (-cos(x-\xi))exp(-(t-s))u(s,\xi)d\xi ds, \qquad (4.2)$$

$t \in [0,2],$ we chose f(t,x) such that the exact solution of (4.2) be u(t,x)=cos(x)exp(-t).

The absolute errors for different values of M and N are reported in Table 2. In this example T=b=2, the numerical results show that the proposed method is suitable for large intervals against some expansion method such that Taylor and Chebyshev expansion methods, the Tau method, the Adomian and homotopy methods, etc. Also for the very large intervals, we can use the introduced method in[9].

(t,x)	Adomian's method with two iterations	Block's method M=N=2
(0.125,0.125)	2.11E-6	3.75E-8
(0.25,0.25)	2.79E-5	2. 61E-8
(0.5,0.5)	2.84E-4	5.51E-7
(0.75,0.75)	-	6.52E-7
(1,1)	1.66E-3	6.72E-7

Table1. Numerical results of Example 1.

(t,x)	M=N=1	M=N=4	M=N=10
(0.125,0.125)	-	4.85E-8	-
(0.25,0.25)	-	3.04E-8	1.41E-10
(0.5,0.5)	6.62E-5	3.81E-9	6.51E-10
(0.75,0.75)	-	1.91E-9	4.10E-10
(1,1)	1.99E-5	3.00E-8	1.11E-10
(1.5,1.5)	2.76E-4	3.71E-8	8.12E-11
(2,2)	2.65E-4	3.62E-8	1.91E-10

Table2. Numerical results of Example 2.

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