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Mehar method for finding exact fuzzy optimal solution of fully fuzzy linear programming problems

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ABSTRACT

There are several methods in the literature to find the fuzzy optimal solution of fully fuzzy linear programming (FFLP) problems. However, in all these methods, it is assumed that the product of two trapezoidal (triangular) fuzzy numbers will also be a trapezoidal (triangular) fuzzy number. Fan *et al*. ("Generalized fuzzy linear programming for decision making under uncertainty: Feasibility of fuzzy solutions and solving approach", Information Sciences, Vol. 241, pp. 12–27, 2013) proposed a method for finding the fuzzy optimal solution of FFLP problems without considering this assumption. In this paper, it is shown that the method proposed by Fan *et al*. (2013) suffer from errors and to overcome these errors, a new method (named as Mehar method) is proposed for solving FFLP problems by modifying the method proposed by Fan *et al*. (2013) . To illustrate the proposed method, some numerical problems are solved.

Keywords: Decision making; Fuzzy parameters; Fuzzy linear programming; Optimization; Uncertainty

1. Introduction

Linear programming is one of the most successively applied operation research techniques. Any linear programming model, representing real world situations, involves a lot of parameters whose values are assigned by experts. However, both experts and decision maker frequently do not precisely know the value of those parameters. Therefore, it is useful to consider the knowledge of experts about the parameters as fuzzy data[9].

The linear programming problems in which all the parameters as well as the variables are represented by fuzzy numbers is known as FFLP problems. FFLP problems can be divided into two categories: (1) FFLP problems with inequality constraints (2) FFLP problems with equality constraints.

Dehghan *et al*.[1] proposed an approach for finding the exact solution of fully fuzzy linear system of equations. Lotfi *et al*.[8] proposed a method to obtain the approximate solution of FFLP problems with equality constraints. Kumar *et al*.[7] proposed a method to find the exact fuzzy optimal solution of FFLP problems with equality constraints having non-negative fuzzy variables and unrestricted fuzzy coefficients. Kaur and Kumar[5] proposed a method to find the exact fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints having unrestricted fuzzy variables. Ezzati *et al*.[2] proposed an algorithm to solve a FFLP by converting it to its equivalent multi-objective linear programming problem. Fan *et al*.[3] pointed out that there is no method in the literature to find the fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints having unrestricted fuzzy co-efficient and proposed a method to find the non-negative fuzzy optimal solution of the same.

All the existing methods[2,5,7,8] except the existing method[3], are proposed by assuming that product of two trapezoidal (triangular) fuzzy numbers is also a trapezoidal (triangular) fuzzy number.

In this paper, the shortcomings of the existing method[3] are pointed out and a new method (named as Mehar method) is proposed to find the non-negative fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints having unrestricted fuzzy co-efficient. Also, it is shown that using the proposed Mehar method, all

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the shortcomings occurring due to using the existing method[3] are resolved.

2. Fan *et al*. method

Fan *et al*.[3] proposed the following method to find the fuzzy optimal solution of fully fuzzy linear programming problem (P1):

$$\text{Maximize (or Minimize)} \quad \left(\sum_{j=1}^{p} \tilde{c}_j \otimes \tilde{x}_j + \sum_{j=p+1}^{n} (-\tilde{c}_j) \otimes \tilde{x}_j \right)$$

Subject to (P1)

$$\sum_{j=1}^{p} \tilde{a}_{ij} \otimes \tilde{x}_j + \sum_{j=p+1}^{n} (-\tilde{a}_{ij}) \otimes \tilde{x}_j \le, =, \ge \ \tilde{b}_i; \quad i = 1, 2, \ldots, m,$$

$\tilde{c}_j, \tilde{a}_{ij}, \tilde{x}_j$ are non-negative triangular fuzzy numbers.

Step 1: Replacing all the fuzzy parameters $\tilde{c}_j$, $\tilde{x}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ of the problem (P1) by the α - cuts $\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right]$, $\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right]$, $\left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right]$ and $\left[(b_i)_a^{\alpha_l}, (b_i)_b^{\alpha_l}\right]$ respectively, obtained corresponding to $\alpha = \alpha_l$, and using the property $\left[(x_i)_a^{\alpha_l}, (x_i)_b^{\alpha_l}\right] \supseteq \left[(x_i)_a^{\alpha_{l-1}}, (x_i)_b^{\alpha_{l-1}}\right]$ of every two α - cuts of a fuzzy number obtained corresponding to $\alpha = \alpha_l$ and $\alpha = \alpha_{l-1}$ respectively, where $\alpha_l < \alpha_{l-1}$, the problem (P1) can be transformed into problem (P2).

$$\text{Maximize (or Minimize)} \quad \left(\sum_{j=1}^{p} \left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] + \sum_{j=p+1}^{n} \left(-\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right]\right)\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \right)$$

Subject to (P2)

$$\sum_{j=1}^{p} \left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] + \sum_{j=p+1}^{n} \left(-\left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right]\right)\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \le, =, \ge \left[(b_i)_a^{\alpha_l}, (b_i)_b^{\alpha_l}\right]; \ \forall i$$

$$\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \supseteq \left[(x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_{l-1}}\right], \left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right] \ge [0,0],$$

$$\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right] \ge [0,0], \left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \ge [0,0].$$

Step 2: Using the property $-[a,b] = [-b,-a]$, the problem (P2) can be transformed into problem (P3).

$$\text{Maximize (or Minimize)} \quad \left(\sum_{j=1}^{p} \left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] + \sum_{j=p+1}^{n} \left[-(c_j)_b^{\alpha_l}, -(c_j)_a^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \right)$$

Subject to (P3)

$$\sum_{j=1}^{p} \left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] + \sum_{j=p+1}^{n} \left[-(a_{ij})_b^{\alpha_l}, -(a_{ij})_a^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \le, =, \ge \left[(b_i)_a^{\alpha_l}, (b_i)_b^{\alpha_l}\right]; \ \forall i$$

$$\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \supseteq \left[(x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_{l-1}}\right], \left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right] \ge [0,0],$$

$$\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right] \ge [0,0], \left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \ge [0,0].$$

Step 3: Using the arithmetic operation,

$$\left[(p_1)_a^{\alpha}, (p_1)_b^{\alpha}\right]\left[(q_1)_a^{\alpha}, (q_1)_b^{\alpha}\right] = \begin{cases} \left[(p_1)_a^{\alpha}(q_1)_a^{\alpha}, (p_1)_b^{\alpha}(q_1)_b^{\alpha}\right]; & \text{if } (p_1)_a^{\alpha} \ge 0, (q_1)_a^{\alpha} \ge 0. \\ \left[(p_1)_a^{\alpha}(q_1)_b^{\alpha}, (p_1)_b^{\alpha}(q_1)_a^{\alpha}\right]; & \text{if } (p_1)_b^{\alpha} \le 0, (q_1)_a^{\alpha} \ge 0. \end{cases}$$

in the objective function and the arithmetic operation

$$\left[(p_1)_a^{\alpha_l},(p_1)_b^{\alpha_l}\right]\left[(q_1)_a^{\alpha_l},(q_1)_b^{\alpha_l}\right]=\begin{cases}\left[(p_1)_b^{\alpha_l}(q_1)_a^{\alpha_l},(p_1)_a^{\alpha_l}(q_1)_b^{\alpha_l}\right]; & \text{if } (p_1)_a^{\alpha_l}\geq 0,(q_1)_a^{\alpha_l}\geq 0.\\ \left[(p_1)_a^{\alpha_l}(q_1)_a^{\alpha_l},(p_1)_b^{\alpha_l}(q_1)_b^{\alpha_l}\right]; & \text{if } (p_1)_b^{\alpha_l}\leq 0,(q_1)_a^{\alpha_l}\geq 0.\end{cases}$$

in the constraints of the problem (P3), it can be transformed into problem (P4).

$$\text{Maximize (or Minimize)}\ \left(\sum_{j=1}^{p}\left[(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l},(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]+\sum_{j=p+1}^{n}\left[-(c_j)_a^{\alpha_l}(x_j)_b^{\alpha_l},-(c_j)_b^{\alpha_l}(x_j)_a^{\alpha_l}\right]\right)$$

Subject to

$$\sum_{j=1}^{p}\left[(a_{ij})_b^{\alpha_l}(x_j)_a^{\alpha_l},(a_{ij})_a^{\alpha_l}(x_j)_b^{\alpha_l}\right]+\sum_{j=p+1}^{n}\left[-(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l},-(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]\leq,=,\geq\left[(b_i)_a^{\alpha_l},(b_i)_b^{\alpha_l}\right];\ \forall i \qquad \text{(P4)}$$

$$\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]\supseteq\left[(x_j)_a^{\alpha_{l-1}},(x_j)_b^{\alpha_{l-1}}\right],\left[(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l}\right]\geq[0,0],$$

$$\left[(c_j)_a^{\alpha_l},(c_j)_b^{\alpha_l}\right]\geq[0,0],\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]\geq[0,0].$$

Step 4: Using the arithmetic operation $\sum[a,b]=\left[\sum a,\sum b\right]$, the problem (P4) can be transformed into problem (P5).

$$\text{Maximize (or Minimize)}\ \left(\left[\sum_{j=1}^{p}(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l},\sum_{j=1}^{p}(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]+\left[\sum_{j=p+1}^{n}-(c_j)_a^{\alpha_l}(x_j)_b^{\alpha_l},\sum_{j=p+1}^{n}-(c_j)_b^{\alpha_l}(x_j)_a^{\alpha_l}\right]\right)$$

Subject to (P5)

$$\left[\sum_{j=1}^{p}(a_{ij})_b^{\alpha_l}(x_j)_a^{\alpha_l},\sum_{j=1}^{p}(a_{ij})_a^{\alpha_l}(x_j)_b^{\alpha_l}\right]+\left[\sum_{j=p+1}^{n}-(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l},\sum_{j=p+1}^{n}-(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]\leq,=,\geq\left[(b_i)_a^{\alpha_l},(b_i)_b^{\alpha_l}\right];\ \forall i$$

$$\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]\supseteq\left[(x_j)_a^{\alpha_{l-1}},(x_j)_b^{\alpha_{l-1}}\right],\left[(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l}\right]\geq[0,0],$$

$$\left[(c_j)_a^{\alpha_l},(c_j)_b^{\alpha_l}\right]\geq[0,0],\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]\geq[0,0].$$

Step 5: Using the arithmetic operation $[a,b]+[c,d]=[a+c,b+d]$, the problem (P5) can be transformed into problem (P6).

$$\text{Maximize (or Minimize)}\ \left(\left[\sum_{j=1}^{p}(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}+\sum_{j=p+1}^{n}-(c_j)_a^{\alpha_l}(x_j)_b^{\alpha_l},\sum_{j=1}^{p}(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}+\sum_{j=p+1}^{n}-(c_j)_b^{\alpha_l}(x_j)_a^{\alpha_l}\right]\right)$$

Subject to (P6)

$$\left[\sum_{j=1}^{p}(a_{ij})_b^{\alpha_l}(x_j)_a^{\alpha_l}+\sum_{j=p+1}^{n}-(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l},\sum_{j=1}^{p}(a_{ij})_a^{\alpha_l}(x_j)_b^{\alpha_l}+\sum_{j=p+1}^{n}-(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]\leq,=,\geq\left[(b_i)_a^{\alpha_l},(b_i)_b^{\alpha_l}\right];\ \forall i$$

$$\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]\supseteq\left[(x_j)_a^{\alpha_{l-1}},(x_j)_b^{\alpha_{l-1}}\right],\left[(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l}\right]\geq[0,0],$$

$$\left[(c_j)_a^{\alpha_l},(c_j)_b^{\alpha_l}\right]\geq[0,0],\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]\geq[0,0].$$

Step 6: Using the properties $[a,b]\leq[c,d]\Rightarrow a\leq c\ \&\ b\leq d$ and $[a,b]\subseteq[c,d]\Rightarrow a\geq c\ \&\ b\leq d$, the problem (P6) can be transformed into problem (P7).

$$\text{Maximize (or Minimize)} \left(\left[\sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(c_j)_a^{\alpha_l} (x_j)_b^{\alpha_l}, \sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} + \sum_{j=p+1}^{n} -(c_j)_b^{\alpha_l} (x_j)_a^{\alpha_l} \right] \right)$$

Subject to

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l};\ \forall i \qquad \text{(P7)}$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_b^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l};\ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_b^{\alpha_l}, (x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}},$$

$(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}; \forall j, \alpha_l \in [0,1]$ are non-negative real numbers.

Step 7: Using the property $\min_{1\le i\le n}[a_i, b_i] = \left[\min_{1\le i\le n} a_i, \min_{1\le i\le n} b_i\right]$, problem (P7) can be transformed into problem (P8).

$$\left(\left[\text{Maximize (or Minimize)} \left(\sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(c_j)_a^{\alpha_l} (x_j)_b^{\alpha_l} \right), \text{Maximize (or Minimize)} \left(\sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} + \sum_{j=p+1}^{n} -(c_j)_b^{\alpha_l} (x_j)_a^{\alpha_l} \right) \right] \right)$$

Subject to

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l};\ \forall i \qquad \text{(P8)}$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_b^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l};\ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_b^{\alpha_l}, (x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}},$$

$(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}; \forall j, \alpha_l \in [0,1]$ are non-negative real numbers.

Step 8: The problem (P8) can be transformed into multi-objective linear programming problem (P9).

$$\text{Maximize (or Minimize)} \left(\sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(c_j)_a^{\alpha_l} (x_j)_b^{\alpha_l} \right)$$

$$\text{Maximize (or Minimize)} \left(\sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} + \sum_{j=p+1}^{n} -(c_j)_b^{\alpha_l} (x_j)_a^{\alpha_l} \right)$$

Subject to

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l};\ \forall i \qquad \text{(P9)}$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_b^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l};\ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_b^{\alpha_l}, (x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}},$$

$(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}; \forall j, \alpha_l \in [0,1]$ are non-negative real numbers.

Step 9: The problem (P9) can be transformed into problem (P10) and problem (P11).

Maximize (or Minimize) $\left(\sum_{j=1}^{p}(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}+\sum_{j=p+1}^{n}-(c_j)_b^{\alpha_l}(x_j)_a^{\alpha_l}\right)$

Subject to

$$\sum_{j=1}^{p}(a_{ij})_a^{\alpha_l}(x_j)_b^{\alpha_l}+\sum_{j=p+1}^{n}-(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\le,=,\ge(b_i)_b^{\alpha_l};\ \forall i \qquad \text{(P10)}$$

$(x_j)_b^{\alpha_l}\ge(x_j)_b^{\alpha_{l-1}}$,

$(c_j)_b^{\alpha_l},(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l},(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l};\forall j,\alpha_l\in[0,1]$ are non-negative real numbers.

Maximize (or Minimize) $\left(\sum_{j=1}^{p}(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}+\sum_{j=p+1}^{n}-(c_j)_a^{\alpha_l}(x_j)_b^{\alpha_l}\right)$

Subject to

$$\sum_{j=1}^{p}(a_{ij})_b^{\alpha_l}(x_j)_a^{\alpha_l}+\sum_{j=p+1}^{n}-(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l}\le,=,\ge(b_i)_a^{\alpha_l};\ \forall i \qquad \text{(P11)}$$

$(x_j)_a^{\alpha_l}\le(x_j)_b^{\alpha_l},(x_j)_a^{\alpha_l}\le(x_j)_a^{\alpha_{l-1}}$,

$(c_j)_a^{\alpha_l},(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l},(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l};\forall j,\alpha_l\in[0,1]$ are non-negative real numbers.

Step 10: Since, the term $\sum_{j=p+1}^{n}-(c_j)_b^{\alpha_l}(x_j)_a^{\alpha_l}$, present in the objective function of problem (P10), is independent from its feasible solution $\{(x_j)_b^{\alpha_l}\}$ and the term $\sum_{j=p+1}^{n}-(c_j)_a^{\alpha_l}(x_j)_b^{\alpha_l}$, present in the objective function of problem (P11), is independent from its feasible solution $\{(x_j)_a^{\alpha_l}\}$. So, the optimal solution of problem (P10) and problem (P11) can be obtained by the equivalent problems (P12) and (P13) respectively.

Maximize (or Minimize) $\left(\sum_{j=1}^{p}(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right)$

Subject to

$$\sum_{j=1}^{p}(a_{ij})_a^{\alpha_l}(x_j)_b^{\alpha_l}+\sum_{j=p+1}^{n}-(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\le,=,\ge(b_i)_b^{\alpha_l};\ \forall i \qquad \text{(P12)}$$

$(x_j)_b^{\alpha_l}\ge(x_j)_b^{\alpha_{l-1}}$,

$(c_j)_b^{\alpha_l},(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l},(x_j)_b^{\alpha_l};\forall j,\alpha_l\in[0,1]$ are non-negative real numbers.

Maximize (or Minimize) $\left(\sum_{j=1}^{p}(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}\right)$

Subject to

$$\sum_{j=1}^{p}(a_{ij})_b^{\alpha_l}(x_j)_a^{\alpha_l} + \sum_{j=p+1}^{n} -(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l} \;;\; \forall i \qquad \text{(P13)}$$

$(x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}$,

$(c_j)_a^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, ;\forall j, \alpha_l \in [0,1]$ are non-negative real numbers.

Step 11: Choose $\alpha_l = \alpha_1 = 1$, and find the optimal solution $\{(x_j)_b^{\alpha_l}\}$ of problem (P12) without considering the constraint $(x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}}$ as well as the optimal solution $\{(x_j)_a^{\alpha_l}\}$ of problem (P13) without considering the constraint $(x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}$.

Step 12: In sequence, put $\alpha_l = \alpha_2, \alpha_l = \alpha_3, ..., \alpha_l = \alpha_q = 0$ in the problem (P12) and problem (P13), where, $1 > \alpha_2 > \alpha_3 > ... > \alpha_q = 0$ and to find the optimal solution of $2(q-1)$ problems by assuming $\alpha_l = \alpha_1 = 1$.

Step 13: Using the optimal solution, obtained in Step 11 and Step 12, the optimal solution of problem (P2) corresponding to α_l is $\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right]$.

Step 14: Using the optimal solution of problem (P2), obtained in Step 13 under different α -cut levels, and regression analysis, find the fuzzy optimal solution of problem (P1).

3. Shortcomings of Fan *et al*. method

It is not genuine to use existing method[3] for finding the solution of problem (P1) due to following reason:

1.If $\left[(p_1)_a^{\alpha}, (p_1)_b^{\alpha}\right]$ and $\left[(q_1)_a^{\alpha}, (q_1)_b^{\alpha}\right]$ are two intervals then

$$\left[(p_1)_a^{\alpha}, (p_1)_b^{\alpha}\right]\left[(q_1)_a^{\alpha}, (q_1)_b^{\alpha}\right] = \begin{cases} \left[(p_1)_a^{\alpha}(q_1)_a^{\alpha}, (p_1)_b^{\alpha}(q_1)_b^{\alpha}\right]; & \text{if } (p_1)_a^{\alpha} \ge 0, (q_1)_a^{\alpha} \ge 0. \\ \left[(p_1)_a^{\alpha}(q_1)_b^{\alpha}, (p_1)_b^{\alpha}(q_1)_a^{\alpha}\right]; & \text{if } (p_1)_b^{\alpha} \le 0, (q_1)_a^{\alpha} \ge 0. \end{cases}$$

However, it is obvious from Step 3 of the existing method [3], described in Section 2, that the authors [3] have used the same multiplication in the objective function to convert the problem (P3) into problem (P4). While, in the constraints the authors [3] have used the mathematically incorrect multiplication.

$$\left[(p_1)_a^{\alpha}, (p_1)_b^{\alpha}\right]\left[(q_1)_a^{\alpha}, (q_1)_b^{\alpha}\right] = \begin{cases} \left[(p_1)_b^{\alpha}(q_1)_a^{\alpha}, (p_1)_a^{\alpha}(q_1)_b^{\alpha}\right]; & \text{if } (p_1)_a^{\alpha} \ge 0, (q_1)_a^{\alpha} \ge 0. \\ \left[(p_1)_a^{\alpha}(q_1)_a^{\alpha}, (p_1)_b^{\alpha}(q_1)_b^{\alpha}\right]; & \text{if } (p_1)_b^{\alpha} \le 0, (q_1)_a^{\alpha} \ge 0. \end{cases}$$

Due to using the mathematically incorrect multiplication, the existing optimal solution [3, pp. 20] of interval valued linear programming problems, shown in Table 1, obtained by the authors [3], are not satisfying its constraints.

α's	Linear Programming Problem	Optimal Solution
1	Maximize $\left[Z_a^1, Z_b^1\right] = [3,3]\left[(x_1)_a^1, (x_1)_b^1\right] + [3,3]\left[(x_2)_a^1, (x_2)_b^1\right]$ Subject to $[1,1]\left[(x_1)_a^1, (x_1)_b^1\right] + [1.4,1.4]\left[(x_2)_a^1, (x_2)_b^1\right] \le [5,5]$ $[1,1]\left[(x_1)_a^1, (x_1)_b^1\right] + [-1,-1]\left[(x_2)_a^1, (x_2)_b^1\right] \le [3,3]$ $\left[(x_1)_a^1, (x_1)_b^1\right] \ge [0,0], \left[(x_2)_a^1, (x_2)_b^1\right] \ge [0,0]$	$\left[(x_{1opt})_a^1, (x_{1opt})_b^1\right] = [3.83, 3.83]$ $\left[(x_{2opt})_a^1, (x_{2opt})_b^1\right] = [0.83, 0.83]$ $\left[\left(Z_a^1\right)_{opt}, \left(Z_b^1\right)_{opt}\right] = [14,14]$

0.9	Maximize $\left[Z_a^{0.9},Z_b^{0.9}\right]=[2.95,3.1]\left[(x_1)_a^{0.9},(x_1)_b^{0.9}\right]+[2.9,3.05]\left[(x_2)_a^{0.9},(x_2)_b^{0.9}\right]$ Subject to $[0.95,1.04]\left[(x_1)_a^{0.9},(x_1)_b^{0.9}\right]+[1.36,1.46]\left[(x_2)_a^{0.9},(x_2)_b^{0.9}\right]\le[4.9,5.2]$ $[0.95,1.05]\left[(x_1)_a^{0.9},(x_1)_b^{0.9}\right]+[-1.05,-0.95]\left[(x_2)_a^{0.9},(x_2)_b^{0.9}\right]\le[2.92,3.1]$ $\left[(x_1)_a^{0.9},(x_1)_b^{0.9}\right]\supseteq\left[(x_{1opt})_a^{1},(x_{1opt})_b^{1}\right],\left[(x_2)_a^{0.9},(x_2)_b^{0.9}\right]\supseteq\left[(x_{2opt})_a^{1},(x_{2opt})_b^{1}\right]$ $\left[(x_1)_a^{0.9},(x_1)_b^{0.9}\right]\ge[0,0],\left[(x_2)_a^{0.9},(x_2)_b^{0.9}\right]\ge[0,0]$	$\left[(x_{1opt})_a^{0.9},(x_{1opt})_b^{0.9}\right]=[3.58,4.17]$ $\left[(x_{2opt})_a^{0.9},(x_{2opt})_b^{0.9}\right]=[0.80,0.91]$ $\left[\left(Z_a^{0.9}\right)_{opt},\left(Z_b^{0.9}\right)_{opt}\right]=[12.9,15.71]$
0.7	Maximize $\left[Z_a^{0.7},Z_b^{0.7}\right]=[2.85,3.3]\left[(x_1)_a^{0.7},(x_1)_b^{0.7}\right]+[2.7,3.15]\left[(x_2)_a^{0.7},(x_2)_b^{0.7}\right]$ Subject to $[0.85,1.12]\left[(x_1)_a^{0.7},(x_1)_b^{0.7}\right]+[1.28,1.58]\left[(x_2)_a^{0.7},(x_2)_b^{0.7}\right]\le[4.7,5.6]$ $[0.85,1.15]\left[(x_1)_a^{0.7},(x_1)_b^{0.7}\right]+[-1.15,-0.85]\left[(x_2)_a^{0.7},(x_2)_b^{0.7}\right]\le[2.76,3.3]$ $\left[(x_1)_a^{0.7},(x_1)_b^{0.7}\right]\supseteq\left[(x_{1opt})_a^{0.9},(x_{1opt})_b^{0.9}\right],\left[(x_2)_a^{0.7},(x_2)_b^{0.7}\right]\supseteq\left[(x_{2opt})_a^{0.9},(x_{2opt})_b^{0.9}\right]$ $\left[(x_1)_a^{0.7},(x_1)_b^{0.7}\right]\ge[0,0],\left[(x_2)_a^{0.7},(x_2)_b^{0.7}\right]\ge[0,0]$	$\left[(x_{1opt})_a^{0.7},(x_{1opt})_b^{0.7}\right]=[3.15,4.96]$ $\left[(x_{2opt})_a^{0.7},(x_{2opt})_b^{0.7}\right]=[0.75,1.08]$ $\left[\left(Z_a^{0.7}\right)_{opt},\left(Z_b^{0.7}\right)_{opt}\right]=[10.98,19.78]$
0.5	Maximize $\left[Z_a^{0.5},Z_b^{0.5}\right]=[2.75,3.5]\left[(x_1)_a^{0.5},(x_1)_b^{0.5}\right]+[2.5,3.25]\left[(x_2)_a^{0.5},(x_2)_b^{0.5}\right]$ Subject to $[0.75,1.2]\left[(x_1)_a^{0.5},(x_1)_b^{0.5}\right]+[1.2,1.7]\left[(x_2)_a^{0.5},(x_2)_b^{0.5}\right]\le[4.5,6]$ $[0.75,1.25]\left[(x_1)_a^{0.5},(x_1)_b^{0.5}\right]+[-1.25,-0.75]\left[(x_2)_a^{0.5},(x_2)_b^{0.5}\right]\le[2.6,3.5]$ $\left[(x_1)_a^{0.5},(x_1)_b^{0.5}\right]\supseteq\left[(x_{1opt})_a^{0.7},(x_{1opt})_b^{0.7}\right],\left[(x_2)_a^{0.5},(x_2)_b^{0.5}\right]\supseteq\left[(x_{2opt})_a^{0.7},(x_{2opt})_b^{0.7}\right]$ $\left[(x_1)_a^{0.5},(x_1)_b^{0.5}\right]\ge[0,0],\left[(x_2)_a^{0.5},(x_2)_b^{0.5}\right]\ge[0,0]$	$\left[(x_{1opt})_a^{0.5},(x_{1opt})_b^{0.5}\right]=[2.77,5.95]$ $\left[(x_{2opt})_a^{0.5},(x_{2opt})_b^{0.5}\right]=[0.69,1.28]$ $\left[\left(Z_a^{0.5}\right)_{opt},\left(Z_b^{0.5}\right)_{opt}\right]=[9.35,24.99]$
0.3	Maximize $\left[Z_a^{0.3},Z_b^{0.3}\right]=[2.65,3.7]\left[(x_1)_a^{0.3},(x_1)_b^{0.3}\right]+[2.3,3.35]\left[(x_2)_a^{0.3},(x_2)_b^{0.3}\right]$ Subject to $[0.65,1.28]\left[(x_1)_a^{0.3},(x_1)_b^{0.3}\right]+[1.12,1.82]\left[(x_2)_a^{0.3},(x_2)_b^{0.3}\right]\le[4.3,6.4]$ $[0.65,1.35]\left[(x_1)_a^{0.3},(x_1)_b^{0.3}\right]+[-1.35,-0.65]\left[(x_2)_a^{0.3},(x_2)_b^{0.3}\right]\le[2.44,3.7]$ $\left[(x_1)_a^{0.3},(x_1)_b^{0.3}\right]\supseteq\left[(x_{1opt})_a^{0.5},(x_{1opt})_b^{0.5}\right],\left[(x_2)_a^{0.3},(x_2)_b^{0.3}\right]\supseteq\left[(x_2)_a^{0.5},(x_2)_b^{0.5}\right]$ $\left[(x_1)_a^{0.3},(x_1)_b^{0.3}\right]\ge[0,0],\left[(x_2)_a^{0.3},(x_2)_b^{0.3}\right]\ge[0,0]$	$\left[(x_{1opt})_a^{0.3},(x_{1opt})_b^{0.3}\right]=[2.45,7.22]$ $\left[(x_{2opt})_a^{0.3},(x_{2opt})_b^{0.3}\right]=[0.64,1.53]$ $\left[\left(Z_a^{0.3}\right)_{opt},\left(Z_b^{0.3}\right)_{opt}\right]=[7.96,31.82]$
0	Maximize $\left[Z_a^{0},Z_b^{0}\right]=[2.5,4]\left[(x_1)_a^{0},(x_1)_b^{0}\right]+[2,3.5]\left[(x_2)_a^{0},(x_2)_b^{0}\right]$ Subject to $[0.5,1.4]\left[(x_1)_a^{0},(x_1)_b^{0}\right]+[1,2]\left[(x_2)_a^{0},(x_2)_b^{0}\right]\le[4,7]$ $[0.5,1.5]\left[(x_1)_a^{0},(x_1)_b^{0}\right]+[-1.5,-0.5]\left[(x_2)_a^{0},(x_2)_b^{0}\right]\le[2.2,4]$ $\left[(x_1)_a^{0},(x_1)_b^{0}\right]\supseteq\left[(x_{1opt})_a^{0.3},(x_{1opt})_b^{0.3}\right],\left[(x_2)_a^{0},(x_2)_b^{0}\right]\supseteq\left[(x_{2opt})_a^{0.3},(x_{2opt})_b^{0.3}\right]$ $\left[(x_1)_a^{0},(x_1)_b^{0}\right]\ge[0,0],\left[(x_2)_a^{0},(x_2)_b^{0}\right]\ge[0,0]$	$\left[(x_{1opt})_a^{0},(x_{1opt})_b^{0}\right]=[2.04,10]$ $\left[(x_{2opt})_a^{0},(x_{2opt})_b^{0}\right]=[0.57,2]$ $\left[\left(Z_a^{0}\right)_{opt},\left(Z_b^{0}\right)_{opt}\right]=[6.24,47]$

Table 1. Existing solutions of problem (P2) [3, pp. 20].

2. Kumar and Kaur[6] have shown that the general form of fully fuzzy linear programming problem is problem (P16), obtained with the help of problem (P17), instead of problem (P14) obtained with the help of problem (P15). However, in the existing method[3], problem (P14) is assumed as the general form of fully fuzzy linear programming problem.

Maximize (or Minimize) $\left(\sum_{j=1}^{p}\tilde{c}_j\tilde{x}_j+\sum_{j=p+1}^{n}(-\tilde{c}_j)\tilde{x}_j\right)$

Subject to (P14)

$$\sum_{j=1}^{p}\tilde{a}_{ij}\tilde{x}_j+\sum_{j=p+1}^{n}(-\tilde{a}_{ij})\tilde{x}_j \le,=,\ge \tilde{b}_i;\quad i=1,2,\ldots,m,$$

$\tilde{c}_j,\tilde{a}_{ij},\tilde{x}_j$ are non-negative fuzzy numbers.

Maximize (or Minimize) $\left(\sum_{j=1}^{p}c_jx_j+\sum_{j=p+1}^{n}(-c_j)x_j\right)$

Subject to (P15)

$$\sum_{j=1}^{p}a_{ij}x_j+\sum_{j=p+1}^{n}(-a_{ij})x_j \le,=,\ge b_i;\quad i=1,2,\ldots,m,$$

c_j,a_{ij},x_j are non-negative real numbers.

Maximize (or Minimize) $(\tilde{Z})$

Subject to

$$\tilde{Z}+\sum_{j=p+1}^{n}\tilde{c}_j\tilde{x}_j=\sum_{j=1}^{p}\tilde{c}_j\tilde{x}_j, \qquad \text{(P16)}$$

$$\sum_{j=1}^{p}\tilde{a}_{ij}\tilde{x}_j \le,=,\ge \tilde{b}_i+\sum_{j=p+1}^{n}\tilde{a}_{ij}\tilde{x}_j;\quad i=1,2,\ldots,m,$$

$\tilde{c}_j,\tilde{a}_{ij},\tilde{x}_j$ are non-negative fuzzy numbers.

Maximize (or Minimize) (Z)

Subject to

$$Z+\sum_{j=p+1}^{n}c_jx_j=\sum_{j=1}^{p}c_jx_j, \qquad \text{(P17)}$$

$$\sum_{j=1}^{p}a_{ij}x_j \le,=,\ge b_i+\sum_{j=p+1}^{n}a_{ij}x_j;\quad i=1,2,\ldots,m,$$

c_j,a_{ij},x_j are non-negative real numbers.

4. Proposed Mehar method

In this section, to resolve the shortcomings, pointed out in Section 3, of the existing method[3], a new method (named as Mehar method) is proposed by modifying the existing method[3] to find the fuzzy optimal solution of problem (P16).

The steps of the proposed Mehar method are as follows:

Step 1: Replacing all the fuzzy parameters $\tilde{Z}$, $\tilde{c}_j$, $\tilde{x}_j$, $\tilde{a}_{ij}$ and $\tilde{b}_i$ of the problem (P16) by their α - cuts $\left[Z_a^{\alpha_l},Z_b^{\alpha_l}\right]$, $\left[(c_j)_a^{\alpha_l},(c_j)_b^{\alpha_l}\right]$, $\left[(x_j)_a^{\alpha_l},(x_j)_b^{\alpha_l}\right]$, $\left[(a_{ij})_a^{\alpha_l},(a_{ij})_b^{\alpha_l}\right]$ and $\left[(b_i)_a^{\alpha_l},(b_i)_b^{\alpha_l}\right]$ respectively, it can be transformed into problem (P18).

Maximize (or Minimize) $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right] + \sum_{j=p+1}^{n}\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] = \sum_{j=1}^{p}\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right], \qquad \text{(P18)}$$

$$\sum_{j=1}^{p}\left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \leq, =, \geq \left[(b_i)_a^{\alpha_l}, (b_i)_b^{\alpha_l}\right] + \sum_{j=p+1}^{n}\left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right]\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right]; \forall i$$

$$\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \supseteq \left[(x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_{l-1}}\right], \left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right] \geq [0,0],$$

$$\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right] \geq [0,0], \left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \geq [0,0].$$

Step 2: Using the arithmetic operation,

$$\left[(p_1)_a^{\alpha}, (p_1)_b^{\alpha}\right]\left[(q_1)_a^{\alpha}, (q_1)_b^{\alpha}\right] = \begin{cases} \left[(p_1)_a^{\alpha}(q_1)_a^{\alpha}, (p_1)_b^{\alpha}(q_1)_b^{\alpha}\right]; & if\,(p_1)_a^{\alpha} \geq 0, (q_1)_a^{\alpha} \geq 0. \\ \left[(p_1)_a^{\alpha}(q_1)_b^{\alpha}, (p_1)_b^{\alpha}(q_1)_a^{\alpha}\right]; & if\,(p_1)_b^{\alpha} \leq 0, (q_1)_a^{\alpha} \geq 0. \end{cases}$$

the problem (P18) can be converted into problem (P19).

Maximize (or Minimize) $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right] + \sum_{j=p+1}^{n}\left[(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right] = \sum_{j=1}^{p}\left[(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right], \qquad \text{(P19)}$$

$$\sum_{j=1}^{p}\left[(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right] \leq, =, \geq \left[(b_i)_a^{\alpha_l}, (b_i)_b^{\alpha_l}\right] + \sum_{j=p+1}^{n}\left[(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]; \forall i$$

$$\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \supseteq \left[(x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_{l-1}}\right], \left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \geq [0,0].$$

Step 3: Using the arithmetic operation $\sum[a,b] = \left[\sum a, \sum b\right]$, the problem (P19) can be transformed into problem (P20).

Maximize (or Minimize) $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right] + \left[\sum_{j=p+1}^{n}(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}, \sum_{j=p+1}^{n}(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right] = \left[\sum_{j=1}^{p}(c_j)_a^{\alpha_l}(x_j)_a^{\alpha_l}, \sum_{j=1}^{p}(c_j)_b^{\alpha_l}(x_j)_b^{\alpha_l}\right], \qquad \text{(P20)}$$

$$\left[\sum_{j=1}^{p}(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l}, \sum_{j=1}^{p}(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right] \leq, =, \geq \left[(b_i)_a^{\alpha_l}, (b_i)_b^{\alpha_l}\right] + \left[\sum_{j=p+1}^{n}(a_{ij})_a^{\alpha_l}(x_j)_a^{\alpha_l}, \sum_{j=p+1}^{n}(a_{ij})_b^{\alpha_l}(x_j)_b^{\alpha_l}\right]; \forall i$$

$$\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \supseteq \left[(x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_{l-1}}\right], \left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \geq [0,0].$$

Step 4: Using the arithmetic operation $[a,b] + [c,d] = [a+c, b+d]$, the problem (P20) can be transformed into problem (P21).

Maximize (or Minimize) $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\left[Z_a^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l}, Z_b^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l}\right] = \left[\sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l}, \sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l}\right], \qquad \text{(P21)}$$

$$\left[\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l}, \sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l}\right] \le,=,\ge \left[(b_i)_a^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l}, (b_i)_b^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l}\right]; \forall i$$

$$\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \supseteq \left[(x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_{l-1}}\right], \left[(a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}\right] \ge [0,0],$$

$$\left[(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}\right] \ge [0,0], \left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right] \ge [0,0].$$

Step 5: Using the properties $[a,b] \le [c,d] \Rightarrow a \le c \ \& \ b \le d$, $[a,b] = [c,d] \Rightarrow a = c \ \& \ b = d$ and $[a,b] \subseteq [c,d] \Rightarrow a \ge c \ \& \ b \le d$, the problem (P21) can be transformed into the problem (P22).

Maximize (or Minimize) $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$Z_a^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} = \sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l},$$

$$Z_b^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} = \sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l}, \qquad \text{(P22)}$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l}; \ \forall i$$

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l}; \ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_b^{\alpha_l}, (x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}},$$

$(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l} \ \forall \ j, \alpha_l \in [0,1]$ are non-negative real numbers.

Step 6: Using the property $\min_{1\le i\le n}[a_i, b_i] = \left[\min_{1\le i\le n} a_i, \min_{1\le i\le n} b_i\right]$, problem (P22) can be transformed into problem (P23).

$\left[\text{Maximize (or Minimize)} \left(Z_a^{\alpha_l}\right), \text{Maximize (or Minimize)}\left(Z_b^{\alpha_l}\right)\right]$

Subject to

$$Z_a^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} = \sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l},$$

$$Z_b^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} = \sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l}, \qquad \text{(P23)}$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l}; \ \forall i$$

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l}; \ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_b^{\alpha_l}, (x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}},$$

$$(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\ \forall\ j, \alpha_l \in [0,1] \text{ are non-negative real numbers.}$$

Step 7: The problem (P23) can be transformed into multi-objective linear programming problem (P24).

Maximize (or Minimize) $\left(Z_a^{\alpha_l}\right)$

Maximize (or Minimize) $\left(Z_b^{\alpha_l}\right)$

Subject to

$$Z_a^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} = \sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l},$$

$$Z_b^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} = \sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l},$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l};\ \forall i \qquad \text{(P24)}$$

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l};\ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_b^{\alpha_l}, (x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}},$$

$$(c_j)_a^{\alpha_l}, (c_j)_b^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\ \forall\ j, \alpha_l \in [0,1] \text{ are non-negative real numbers.}$$

Step 8: The problem (P24) can be transformed into problem (P25) and problem (P26).

Maximize (or Minimize) $\left(Z_b^{\alpha_l}\right)$

Subject to

$$Z_b^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l} = \sum_{j=1}^{p} (c_j)_b^{\alpha_l} (x_j)_b^{\alpha_l}, \qquad \text{(P25)}$$

$$\sum_{j=1}^{p} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l} \le,=,\ge (b_i)_b^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_b^{\alpha_l} (x_j)_b^{\alpha_l};\ \forall i$$

$$(x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}}, (c_j)_b^{\alpha_l}, (a_{ij})_b^{\alpha_l}, (x_j)_b^{\alpha_l}\ \forall\ j, \alpha_l \in [0,1] \text{ are non-negative real numbers.}$$

Maximize (or Minimize) $\left(Z_a^{\alpha_l}\right)$

Subject to

$$Z_a^{\alpha_l} + \sum_{j=p+1}^{n} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l} = \sum_{j=1}^{p} (c_j)_a^{\alpha_l} (x_j)_a^{\alpha_l}, \qquad \text{(P26)}$$

$$\sum_{j=1}^{p} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l} \le,=,\ge (b_i)_a^{\alpha_l} + \sum_{j=p+1}^{n} (a_{ij})_a^{\alpha_l} (x_j)_a^{\alpha_l};\ \forall i$$

$$(x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}, (c_j)_a^{\alpha_l}, (a_{ij})_a^{\alpha_l}, (x_j)_a^{\alpha_l}\ \forall\ j, \alpha_l \in [0,1] \text{ are non-negative real numbers.}$$

Step 9: Choose $\alpha_l = \alpha_1 = 1$, and find the optimal solution $\left\{(x_j)_b^{\alpha_l}\right\}$ of problem (P25) without considering the constraint $(x_j)_b^{\alpha_l} \ge (x_j)_b^{\alpha_{l-1}}$ and find the optimal solution $\left\{(x_j)_a^{\alpha_l}\right\}$ of problem (P26) without considering the constraint $(x_j)_a^{\alpha_l} \le (x_j)_a^{\alpha_{l-1}}$.

Step 10: In sequence, put $\alpha_l = \alpha_2,\ \alpha_l = \alpha_3,\ ..., \alpha_l = \alpha_q = 0$ in the problem (P25) and problem (P26), where, $1 > \alpha_2 > \alpha_3 > ... > \alpha_q = 0$ and to find the optimal solution of $2(q-1)$ problems by assuming $\alpha_l = \alpha_1 = 1$.

Step 11: Using the optimal solution, obtained in Step 9 and Step 10, the optimal solution of problem (P18) corresponding to α_l is $\left[(x_j)_a^{\alpha_l}, (x_j)_b^{\alpha_l}\right]$.

Step 12: Using the optimal solution of problem (P18), obtained in Step 11 under different α -cut levels, and regression analysis, find the fuzzy optimal solution of problem (P16).

5. Fuzzy optimal solution of some fully fuzzy linear programming problems

In this section, the fuzzy optimal solution of the existing problem[3] and the existing problem[6], are obtained by the proposed Mehar method.

5.1 Fuzzy optimal solution of existing problem

Fan *et al*.[3] solved the problem (P27) to illustrate their proposed method which is obtained by replacing the crisp parameters $c_1 = 3, c_2 = 3, a_{11} = 1, a_{12} = 1.5, a_{21} = 1, a_{22} = -1, b_1 = 5$ and $b_2 = 3$ of problem (P28) by the fuzzy parameters. However, as discussed in Section 3, it is not genuine to use the existing form of fully fuzzy linear programming problem (P27), obtained from problem (P28). So, in this section, instead of using the general form of fully fuzzy linear programming problem (P27), obtained from problem (P28), the general form of fully fuzzy linear programming problem (P29), obtained from crisp linear programming problem (P30), is solved by the proposed Mehar method.

Maximize $\left(\tilde{Z} = (2.5,3,4)\tilde{x}_1 + (2,3,3.5)\tilde{x}_2\right)$

Subject to

$$(0.5,1,1.4)\tilde{x}_1 + (1,1.4,2)\tilde{x}_2 \le (4,5,7), \qquad \text{(P27)}$$
$$(0.5,1,1.5)\tilde{x}_1 + (-(0.5,1,1.5))\tilde{x}_2 \le (2.2,3,4),$$
$\tilde{x}_1, \tilde{x}_2$ are non-negative fuzzy numbers.

Maximize $\left(Z = 3x_1 + 3x_2\right)$

Subject to

$$x_1 + 1.5x_2 \le 5, \qquad \text{(P28)}$$
$$x_1 + (-1)x_2 \le 3,$$
$$x_1, x_2 \ge 0.$$

Maximize $\left(\tilde{Z}\right)$

Subject to

$$\tilde{Z} = (2.5,3,4)\tilde{x}_1 + (2,3,3.5)\tilde{x}_2,$$
$$(0.5,1,1.4)\tilde{x}_1 + (1,1.4,2)\tilde{x}_2 \le (4,5,7), \qquad \text{(P29)}$$
$$(0.5,1,1.5)\tilde{x}_1 \le (2.2,3,4) + (0.5,1,1.5)\tilde{x}_2,$$
$\tilde{x}_1, \tilde{x}_2$ are non-negative fuzzy numbers.

Maximize $\left(Z\right)$

Subject to

$$Z = 3x_1 + 3x_2,$$
$$x_1 + 1.5x_2 \le 5, \qquad \text{(P30)}$$
$$x_1 \le 3 + x_2,$$
$$x_1, x_2 \ge 0.$$

Step 1: Replacing all the fuzzy parameters $\tilde{Z}$, $\tilde{c}_1=(2.5,3,4)$, $\tilde{c}_2=(2,3,3.5)$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{a}_{11}=(0.5,1,1.4)$, $\tilde{a}_{12}=(1,1.4,2)$, $\tilde{a}_{21}=(0.5,1,1.5)$, $\tilde{a}_{22}=(0.5,1,1.5)$, $\tilde{b}_1=(4,5,7)$ and $\tilde{b}_2=(2.2,3,4)$ of the problem (P29) by their α - cuts $\left[Z_a^{\alpha_l},Z_b^{\alpha_l}\right]$, $\left[2.5+0.5\alpha_l,4-\alpha_l\right]$, $\left[2+\alpha_l,3.5-0.5\alpha_l\right]$, $\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]$, $\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right]$, $\left[0.5+0.5\alpha_l,1.4-0.4\alpha_l\right]$, $\left[1+0.4\alpha_l,2-0.6\alpha_l\right]$, $\left[0.5+0.5\alpha_l,1.5-0.5\alpha_l\right]$, $\left[0.5+0.5\alpha_l,1.5-0.5\alpha_l\right]$, $\left[4+\alpha_l,7-2\alpha_l\right]$ and $\left[2.2+0.8\alpha_l,4-\alpha_l\right]$ respectively, it can be transformed into problem (P31).

$$\text{Maximize } \left(\left[Z_a^{\alpha_l},Z_b^{\alpha_l}\right]\right)$$

Subject to

$$\left[Z_a^{\alpha_l},Z_b^{\alpha_l}\right]=\left[2.5+0.5\alpha_l,4-\alpha_l\right]\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]+\left[2+\alpha_l,3.5-0.5\alpha_l\right]\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right], \qquad \text{(P31)}$$

$$\left[0.5+0.5\alpha_l,1.4-0.4\alpha_l\right]\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]+\left[1+0.4\alpha_l,2-0.6\alpha_l\right]\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right]\le\left[4+\alpha_l,7-2\alpha_l\right],$$

$$\left[0.5+0.5\alpha_l,1.5-0.5\alpha_l\right]\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]\le\left[2.2+0.8\alpha_l,4-\alpha_l\right]+\left[0.5+0.5\alpha_l,1.5-0.5\alpha_l\right]\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right],$$

$$\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]\supseteq\left[(x_1)_a^{\alpha_{l-1}},(x_1)_b^{\alpha_{l-1}}\right],\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right]\supseteq\left[(x_2)_a^{\alpha_{l-1}},(x_2)_b^{\alpha_{l-1}}\right],$$

$$\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]\ge[0,0],\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right]\ge[0,0];\ \forall\alpha_l\in[0,1].$$

Step 2: Using the arithmetic operation,

$$\left[(p_1)_a^{\alpha},(p_1)_b^{\alpha}\right]\left[(q_1)_a^{\alpha},(q_1)_b^{\alpha}\right]=\begin{cases}\left[(p_1)_a^{\alpha}(q_1)_a^{\alpha},(p_1)_b^{\alpha}(q_1)_b^{\alpha}\right]; & \text{if } (p_1)_a^{\alpha}\ge 0,(q_1)_a^{\alpha}\ge 0,\\ \left[(p_1)_a^{\alpha}(q_1)_b^{\alpha},(p_1)_b^{\alpha}(q_1)_a^{\alpha}\right]; & \text{if } (p_1)_b^{\alpha}\le 0,(q_1)_a^{\alpha}\ge 0.\end{cases}$$

the problem (P31) can be transformed into problem (P32).

$$\text{Maximize } \left(\left[Z_a^{\alpha_l},Z_b^{\alpha_l}\right]\right)$$

Subject to

$$\left[Z_a^{\alpha_l},Z_b^{\alpha_l}\right]=\left[\left(2.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}+\left(2+\alpha_l\right)(x_2)_a^{\alpha_l},\left(4-\alpha_l\right)(x_1)_b^{\alpha_l}+\left(3.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l}\right],$$

$$\left[\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}+\left(1+0.4\alpha_l\right)(x_2)_a^{\alpha_l},\left(1.4-0.4\alpha_l\right)(x_1)_b^{\alpha_l}+\left(2-0.6\alpha_l\right)(x_2)_b^{\alpha_l}\right]\le\left[4+\alpha_l,7-2\alpha_l\right], \quad \text{(P32)}$$

$$\left[\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l},\left(1.5-0.5\alpha_l\right)(x_1)_b^{\alpha_l}\right]\le\left[\left(2.2+0.8\alpha_l\right)+\left(0.5+0.5\alpha_l\right)(x_2)_a^{\alpha_l},\left(4-\alpha_l\right)+\left(1.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l}\right],$$

$$\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]\supseteq\left[(x_1)_a^{\alpha_{l-1}},(x_1)_b^{\alpha_{l-1}}\right],\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right]\supseteq\left[(x_2)_a^{\alpha_{l-1}},(x_2)_b^{\alpha_{l-1}}\right],$$

$$\left[(x_1)_a^{\alpha_l},(x_1)_b^{\alpha_l}\right]\ge[0,0],\left[(x_2)_a^{\alpha_l},(x_2)_b^{\alpha_l}\right]\ge[0,0];\ \forall\alpha_l\in[0,1].$$

Step 3: Using the properties $[a,b]\le[c,d]\Rightarrow a\le c\ \&\ b\le d$, $[a,b]=[c,d]\Rightarrow a=c\ \&\ b=d$ and $[a,b]\subseteq[c,d]\Rightarrow a\ge c\ \&\ b\le d$, the problem (P32) can be transformed into the problem (P33).

Maximize $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\begin{aligned}
&Z_a^{\alpha_l} = \left(2.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l} + \left(2+\alpha_l\right)(x_2)_a^{\alpha_l},\\
&Z_b^{\alpha_l} = \left(4-\alpha_l\right)(x_1)_b^{\alpha_l} + \left(3.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},\\
&\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l} + \left(1+0.4\alpha_l\right)(x_2)_a^{\alpha_l} \le 4+\alpha_l,\\
&\left(1.4-0.4\alpha_l\right)(x_1)_b^{\alpha_l} + \left(2-0.6\alpha_l\right)(x_2)_b^{\alpha_l} \le 7-2\alpha_l,\\
&\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l} \le \left(2.2+0.8\alpha_l\right) + \left(0.5+0.5\alpha_l\right)(x_2)_a^{\alpha_l},\\
&\left(1.5-0.5\alpha_l\right)(x_1)_b^{\alpha_l} \le \left(4-\alpha_l\right) + \left(1.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},\\
&(x_1)_a^{\alpha_l} \le (x_1)_b^{\alpha_l}, (x_2)_a^{\alpha_l} \le (x_2)_b^{\alpha_l}, (x_1)_a^{\alpha_l} \le (x_1)_a^{\alpha_{l-1}}, (x_2)_a^{\alpha_l} \le (x_2)_a^{\alpha_{l-1}}, (x_1)_b^{\alpha_l} \ge (x_1)_b^{\alpha_{l-1}},\\
&(x_2)_b^{\alpha_l} \ge (x_2)_b^{\alpha_{l-1}}, (x_1)_a^{\alpha_l} \ge 0, (x_1)_b^{\alpha_l} \ge 0, (x_2)_a^{\alpha_l} \ge 0, (x_2)_b^{\alpha_l} \ge 0;\ \forall \alpha_l \in [0,1].
\end{aligned} \tag{P33}$$

Step 4: Using the property $\min_{1\le i\le n}\left[a_i, b_i\right] = \left[\min_{1\le i\le n} a_i, \min_{1\le i\le n} b_i\right]$, problem (P33) can be transformed into problem (P34).

$\left[\text{Maximize}\left(Z_a^{\alpha_l}\right), \text{Maximize}\left(Z_b^{\alpha_l}\right)\right]$

Subject to

$$\begin{aligned}
&Z_a^{\alpha_l} = \left(2.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l} + \left(2+\alpha_l\right)(x_2)_a^{\alpha_l},\\
&Z_b^{\alpha_l} = \left(4-\alpha_l\right)(x_1)_b^{\alpha_l} + \left(3.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},\\
&\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l} + \left(1+0.4\alpha_l\right)(x_2)_a^{\alpha_l} \le 4+\alpha_l,\\
&\left(1.4-0.4\alpha_l\right)(x_1)_b^{\alpha_l} + \left(2-0.6\alpha_l\right)(x_2)_b^{\alpha_l} \le 7-2\alpha_l,\\
&\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l} \le \left(2.2+0.8\alpha_l\right) + \left(0.5+0.5\alpha_l\right)(x_2)_a^{\alpha_l},\\
&\left(1.5-0.5\alpha_l\right)(x_1)_b^{\alpha_l} \le \left(4-\alpha_l\right) + \left(1.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},\\
&(x_1)_a^{\alpha_l} \le (x_1)_b^{\alpha_l}, (x_2)_a^{\alpha_l} \le (x_2)_b^{\alpha_l}, (x_1)_a^{\alpha_l} \le (x_1)_a^{\alpha_{l-1}}, (x_2)_a^{\alpha_l} \le (x_2)_a^{\alpha_{l-1}}, (x_1)_b^{\alpha_l} \ge (x_1)_b^{\alpha_{l-1}},\\
&(x_2)_b^{\alpha_l} \ge (x_2)_b^{\alpha_{l-1}}, (x_1)_a^{\alpha_l} \ge 0, (x_1)_b^{\alpha_l} \ge 0, (x_2)_a^{\alpha_l} \ge 0, (x_2)_b^{\alpha_l} \ge 0;\ \forall \alpha_l \in [0,1].
\end{aligned} \tag{P34}$$

Step 5: The problem (P34) can be transformed into multi-objective linear programming problem (P35).

Maximize $\left(Z_a^{\alpha_l}\right)$

Maximize $\left(Z_b^{\alpha_l}\right)$

Subject to

$$Z_a^{\alpha_l}=\left(2.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}+\left(2+\alpha_l\right)(x_2)_a^{\alpha_l},$$
$$Z_b^{\alpha_l}=\left(4-\alpha_l\right)(x_1)_b^{\alpha_l}+\left(3.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},$$
$$\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}+\left(1+0.4\alpha_l\right)(x_2)_a^{\alpha_l}\le 4+\alpha_l,$$
$$\left(1.4-0.4\alpha_l\right)(x_1)_b^{\alpha_l}+\left(2-0.6\alpha_l\right)(x_2)_b^{\alpha_l}\le 7-2\alpha_l, \qquad \text{(P35)}$$
$$\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}\le\left(2.2+0.8\alpha_l\right)+\left(0.5+0.5\alpha_l\right)(x_2)_a^{\alpha_l},$$
$$\left(1.5-0.5\alpha_l\right)(x_1)_b^{\alpha_l}\le\left(4-\alpha_l\right)+\left(1.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},$$
$$(x_1)_a^{\alpha_l}\le(x_1)_b^{\alpha_l},(x_2)_a^{\alpha_l}\le(x_2)_b^{\alpha_l},(x_1)_a^{\alpha_l}\le(x_1)_a^{\alpha_{l-1}},(x_2)_a^{\alpha_l}\le(x_2)_a^{\alpha_{l-1}},\ (x_1)_b^{\alpha_l}\ge(x_1)_b^{\alpha_{l-1}},$$
$$(x_2)_b^{\alpha_l}\ge(x_2)_b^{\alpha_{l-1}},(x_1)_a^{\alpha_l}\ge 0,(x_1)_b^{\alpha_l}\ge 0,(x_2)_a^{\alpha_l}\ge 0,(x_2)_b^{\alpha_l}\ge 0;\ \forall\alpha_l\in[0,1].$$

Step 6: The problem (P35) can be transformed into the problems (P36) and (P37).

Maximize $\left(Z_b^{\alpha_l}\right)$

Subject to

$$Z_b^{\alpha_l}=\left(4-\alpha_l\right)(x_1)_b^{\alpha_l}+\left(3.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},$$
$$\left(1.4-0.4\alpha_l\right)(x_1)_b^{\alpha_l}+\left(2-0.6\alpha_l\right)(x_2)_b^{\alpha_l}\le 7-2\alpha_l, \qquad \text{(P36)}$$
$$\left(1.5-0.5\alpha_l\right)(x_1)_b^{\alpha_l}\le\left(4-\alpha_l\right)+\left(1.5-0.5\alpha_l\right)(x_2)_b^{\alpha_l},$$
$$(x_1)_b^{\alpha_l}\ge(x_1)_b^{\alpha_{l-1}},(x_2)_b^{\alpha_l}\ge(x_2)_b^{\alpha_{l-1}},(x_1)_b^{\alpha_l}\ge 0,(x_2)_b^{\alpha_l}\ge 0;\ \forall\alpha_l\in[0,1].$$

Maximize $\left(Z_a^{\alpha_l}\right)$

Subject to

$$Z_a^{\alpha_l}=\left(2.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}+\left(2+\alpha_l\right)(x_2)_a^{\alpha_l},$$
$$\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}+\left(1+0.4\alpha_l\right)(x_2)_a^{\alpha_l}\le 4+\alpha_l, \qquad \text{(P37)}$$
$$\left(0.5+0.5\alpha_l\right)(x_1)_a^{\alpha_l}\le\left(2.2+0.8\alpha_l\right)+\left(0.5+0.5\alpha_l\right)(x_2)_a^{\alpha_l},$$
$$(x_1)_a^{\alpha_l}\le(x_1)_a^{\alpha_{l-1}},(x_2)_a^{\alpha_l}\le(x_2)_a^{\alpha_{l-1}},(x_1)_a^{\alpha_l}\ge 0,(x_2)_a^{\alpha_l}\ge 0;\ \forall\alpha_l\in[0,1].$$

Step 7: The optimal solution $\left\{(x_j)_b^{\alpha_l}\right\}$ of problem (P36) without considering the constraint $(x_1)_b^{\alpha_l}\ge(x_1)_b^{\alpha_{l-1}},(x_2)_b^{\alpha_l}\ge(x_2)_b^{\alpha_{l-1}}$ and the optimal solution $\left\{(x_j)_a^{\alpha_l}\right\}$ of problem (P37) without considering the constraint $(x_1)_a^{\alpha_l}\le(x_1)_a^{\alpha_{l-1}},(x_2)_a^{\alpha_l}\le(x_2)_a^{\alpha_{l-1}}$ for $\alpha_l=\alpha_1=1$ is shown in **Table 2**.

α's	Linear Programming Problem	Optimal Solution

1	Maximize Z_b^1 Subject to $Z_b^1 = 3(x_1)_b^1 + 3(x_2)_b^1$ $(x_1)_b^1 + 1.4(x_2)_b^1 \le 5$ $(x_1)_b^1 \le 3 + (x_2)_b^1$ $(x_1)_b^1 \ge 0, (x_2)_b^1 \ge 0$	$\left[(x_{1opt})_a^1, (x_{1opt})_b^1\right] = [3.83, 3.83]$ $\left[(x_{2opt})_a^1, (x_{2opt})_b^1\right] = [0.83, 0.83]$ $\left[\left(Z_a^1\right)_{opt}, \left(Z_b^1\right)_{opt}\right] = [14, 14]$
	Maximize Z_a^1 Subject to $Z_a^1 = 3(x_1)_a^1 + 3(x_2)_a^1$ $(x_1)_a^1 + 1.4(x_2)_a^1 \le 5$ $(x_1)_a^1 \le 3 + (x_2)_a^1$ $(x_1)_a^1 \ge 0, (x_2)_a^1 \ge 0$	

Table 2. Solutions of problem (P36) and (P37) at $\alpha_l = \alpha_1 = 1$.

Step 8: The optimal solution $\left\{(x_j)_b^{\alpha_l}\right\}$ of problem (P36) and the optimal solution $\left\{(x_j)_a^{\alpha_l}\right\}$ of problem (P37) for $\alpha_l = \alpha_2 = 0.9$ is shown in Table 3.

Table 3 Solutions of problem (P36) and problem (P37) at $\alpha_l = \alpha_2 = 0.9$.

$\alpha's$	Linear Programming Problem	Optimal Solution
0.9	Maximize $Z_b^{0.9}$ Subject to $Z_b^{0.9} = 3.1(x_1)_b^{0.9} + 3.05(x_2)_b^{0.9}$ $1.04(x_1)_b^{0.9} + 1.46(x_2)_b^{0.9} \le 5.2$ $1.05(x_1)_b^{0.9} \le 3.1 + 1.05(x_2)_b^{0.9}$ $(x_1)_b^{0.9} \ge (x_{1opt})_b^1, (x_2)_b^{0.9} \ge (x_{2opt})_b^1$ $(x_1)_b^{0.9} \ge 0, (x_2)_b^{0.9} \ge 0$	Infeasible
	Maximize $Z_a^{0.9}$ Subject to $Z_a^{0.9} = 2.95(x_1)_a^{0.9} + 2.9(x_2)_a^{0.9}$ $0.95(x_1)_a^{0.9} + 1.36(x_2)_a^{0.9} \le 4.9$ $0.95(x_1)_a^{0.9} \le 2.92 + 0.95(x_2)_a^{0.9}$ $(x_1)_a^{0.9} \le (x_2)_a^{0.9}, (x_2)_a^{0.9} \le (x_{2opt})_a^1$ $(x_1)_a^{0.9} \ge 0, (x_2)_a^{0.9} \ge 0$	

5.2 Fuzzy optimal solution of the existing problem

Kumar and Kaur[6] solved the problem (P38) to illustrate their proposed method by assuming that product of two triangular/trapezoidal fuzzy numbers will also be a triangular/trapezoidal fuzzy number. In this section, the exact fuzzy optimal solution of the same problem is obtained by the proposed Mehar method.

Maximize $\left(\tilde{Z}\right)$

Subject to

$$\left(0,10,30,40\right)\left(\tilde{x}_{11}+\tilde{x}_{12}+\tilde{x}_{13}\right)+\left(10,20,30,40\right)\left(\tilde{x}_{21}+\tilde{x}_{22}+\tilde{x}_{23}\right)+\left(10,22,22,34\right)\left(\tilde{x}_{31}+\tilde{x}_{32}+\tilde{x}_{33}\right)$$
$$+\left(8,10,14,16\right)\left(\tilde{x}_{41}+\tilde{x}_{42}+\tilde{x}_{43}\right)=\tilde{Z}+\left(0,8,8,16\right)\left(\tilde{x}_{11}+\tilde{x}_{21}+\tilde{x}_{31}+\tilde{x}_{41}\right)$$
$$+\left(5,9,10,12\right)\left(\tilde{x}_{12}+\tilde{x}_{22}+\tilde{x}_{32}+\tilde{x}_{42}\right)+\left(0,5,9,14\right)\left(\tilde{x}_{13}+\tilde{x}_{23}+\tilde{x}_{33}+\tilde{x}_{43}\right),$$
$$\tilde{x}_{11}\geq 0.40\left(\tilde{x}_{11}+\tilde{x}_{12}+\tilde{x}_{13}\right),$$
$$\tilde{x}_{21}\geq 0.20\left(\tilde{x}_{21}+\tilde{x}_{22}+\tilde{x}_{23}\right),$$
$$\tilde{x}_{31}\geq 0.50\left(\tilde{x}_{31}+\tilde{x}_{32}+\tilde{x}_{33}\right), \qquad \text{(P38)}$$
$$\tilde{x}_{12}\leq 0.20\left(\tilde{x}_{11}+\tilde{x}_{12}+\tilde{x}_{13}\right),$$
$$\tilde{x}_{22}\leq 0.40\left(\tilde{x}_{21}+\tilde{x}_{22}+\tilde{x}_{23}\right),$$
$$\tilde{x}_{32}\leq 0.10\left(\tilde{x}_{31}+\tilde{x}_{32}+\tilde{x}_{33}\right),$$
$$\left(\tilde{x}_{11}+\tilde{x}_{21}+\tilde{x}_{31}+\tilde{x}_{41}\right)\leq\left(100,150,250,300\right),$$
$$\left(\tilde{x}_{12}+\tilde{x}_{22}+\tilde{x}_{32}+\tilde{x}_{42}\right)\leq\left(160,180,220,240\right),$$
$$\left(\tilde{x}_{13}+\tilde{x}_{23}+\tilde{x}_{33}+\tilde{x}_{43}\right)\leq\left(50,100,200,250\right),$$

$\tilde{x}_{ij};\ i=1,2,3,4;\ j=1,2,3$ are non-negative trapezoidal fuzzy numbers.

Step 1: Replacing all the fuzzy numbers of the problem (P38) by their α – cuts, it can be transformed into problem (P39).

Maximize $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\left[10\alpha_l, 40-10\alpha_l\right]\left(\left[(x_{11})_a^{\alpha_l}, (x_{11})_b^{\alpha_l}\right]+\left[(x_{12})_a^{\alpha_l}, (x_{12})_b^{\alpha_l}\right]+\left[(x_{13})_a^{\alpha_l}, (x_{13})_b^{\alpha_l}\right]\right)+\left[10+10\alpha_l, 40-10\alpha_l\right]\left(\left[(x_{21})_a^{\alpha_l}, (x_{21})_b^{\alpha_l}\right]\right.$$
$$\left.+\left[(x_{22})_a^{\alpha_l}, (x_{22})_b^{\alpha_l}\right]+\left[(x_{23})_a^{\alpha_l}, (x_{23})_b^{\alpha_l}\right]\right)+\left[10+12\alpha_l, 34-12\alpha_l\right]\left(\left[(x_{31})_a^{\alpha_l}, (x_{31})_b^{\alpha_l}\right]+\left[(x_{32})_a^{\alpha_l}, (x_{32})_b^{\alpha_l}\right]+\left[(x_{33})_a^{\alpha_l}, (x_{33})_b^{\alpha_l}\right]\right)$$
$$+\left[8+2\alpha_l, 16-2\alpha_l\right]\left(\left[(x_{41})_a^{\alpha_l}, (x_{41})_b^{\alpha_l}\right]+\left[(x_{42})_a^{\alpha_l}, (x_{42})_b^{\alpha_l}\right]+\left[(x_{43})_a^{\alpha_l}, (x_{43})_b^{\alpha_l}\right]\right)=\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]+\left[8\alpha_l, 16-8\alpha_l\right]\left(\left[(x_{11})_a^{\alpha_l}, (x_{11})_b^{\alpha_l}\right]\right.$$
$$\left.+\left[(x_{21})_a^{\alpha_l}, (x_{21})_b^{\alpha_l}\right]+\left[(x_{31})_a^{\alpha_l}, (x_{31})_b^{\alpha_l}\right]+\left[(x_{41})_a^{\alpha_l}, (x_{41})_b^{\alpha_l}\right]\right)+\left[5+4\alpha_l, 12-2\alpha_l\right]\left(\left[(x_{12})_a^{\alpha_l}, (x_{12})_b^{\alpha_l}\right]+\left[(x_{22})_a^{\alpha_l}, (x_{22})_b^{\alpha_l}\right]\right.$$
$$\left.+\left[(x_{32})_a^{\alpha_l}, (x_{32})_b^{\alpha_l}\right]+\left[(x_{42})_a^{\alpha_l}, (x_{42})_b^{\alpha_l}\right]\right)+\left[5\alpha_l, 14-5\alpha_l\right]\left(\left[(x_{13})_a^{\alpha_l}, (x_{13})_b^{\alpha_l}\right]+\left[(x_{23})_a^{\alpha_l}, (x_{23})_b^{\alpha_l}\right]+\left[(x_{33})_a^{\alpha_l}, (x_{33})_b^{\alpha_l}\right]\right.$$
$$\left.+\left[(x_{43})_a^{\alpha_l}, (x_{43})_b^{\alpha_l}\right]\right),$$
$$\left[(x_{11})_a^{\alpha_l}, (x_{11})_b^{\alpha_l}\right]\geq 0.40\left(\left[(x_{11})_a^{\alpha_l}, (x_{11})_b^{\alpha_l}\right]+\left[(x_{12})_a^{\alpha_l}, (x_{12})_b^{\alpha_l}\right]+\left[(x_{13})_a^{\alpha_l}, (x_{13})_b^{\alpha_l}\right]\right),$$
$$\left[(x_{21})_a^{\alpha_l}, (x_{21})_b^{\alpha_l}\right]\geq 0.20\left(\left[(x_{21})_a^{\alpha_l}, (x_{21})_b^{\alpha_l}\right]+\left[(x_{22})_a^{\alpha_l}, (x_{22})_b^{\alpha_l}\right]+\left[(x_{23})_a^{\alpha_l}, (x_{23})_b^{\alpha_l}\right]\right), \qquad \text{(P39)}$$
$$\left[(x_{31})_a^{\alpha_l}, (x_{31})_b^{\alpha_l}\right]\geq 0.50\left(\left[(x_{31})_a^{\alpha_l}, (x_{31})_b^{\alpha_l}\right]+\left[(x_{32})_a^{\alpha_l}, (x_{32})_b^{\alpha_l}\right]+\left[(x_{33})_a^{\alpha}, (x_{33})_b^{\alpha_l}\right]\right),$$
$$\left[(x_{12})_a^{\alpha_l}, (x_{12})_b^{\alpha_l}\right]\leq 0.20\left(\left[(x_{11})_a^{\alpha_l}, (x_{11})_b^{\alpha_l}\right]+\left[(x_{12})_a^{\alpha_l}, (x_{12})_b^{\alpha_l}\right]+\left[(x_{13})_a^{\alpha_l}, (x_{13})_b^{\alpha_l}\right]\right),$$
$$\left[(x_{22})_a^{\alpha_l}, (x_{22})_b^{\alpha_l}\right]\leq 0.40\left(\left[(x_{21})_a^{\alpha_l}, (x_{21})_b^{\alpha_l}\right]+\left[(x_{22})_a^{\alpha_l}, (x_{22})_b^{\alpha_l}\right]+\left[(x_{23})_a^{\alpha_l}, (x_{23})_b^{\alpha_l}\right]\right),$$
$$\left[(x_{32})_a^{\alpha_l}, (x_{32})_b^{\alpha_l}\right]\leq 0.10\left(\left[(x_{31})_a^{\alpha_l}, (x_{31})_b^{\alpha_l}\right]+\left[(x_{32})_a^{\alpha_l}, (x_{32})_b^{\alpha_l}\right]+\left[(x_{33})_a^{\alpha_l}, (x_{33})_b^{\alpha_l}\right]\right),$$
$$\left(\left[(x_{11})_a^{\alpha_l}, (x_{11})_b^{\alpha_l}\right]+\left[(x_{21})_a^{\alpha_l}, (x_{21})_b^{\alpha_l}\right]+\left[(x_{31})_a^{\alpha_l}, (x_{31})_b^{\alpha_l}\right]+\left[(x_{41})_a^{\alpha_l}, (x_{41})_b^{\alpha_l}\right]\right)\leq\left[100+50\alpha_l, 300-50\alpha_l\right],$$
$$\left(\left[(x_{12})_a^{\alpha_l}, (x_{12})_b^{\alpha_l}\right]+\left[(x_{22})_a^{\alpha_l}, (x_{22})_b^{\alpha_l}\right]+\left[(x_{32})_a^{\alpha_l}, (x_{32})_b^{\alpha_l}\right]+\left[(x_{42})_a^{\alpha_l}, (x_{42})_b^{\alpha_l}\right]\right)\leq\left[160+20\alpha_l, 240-20\alpha_l\right],$$
$$\left(\left[(x_{13})_a^{\alpha_l}, (x_{13})_b^{\alpha_l}\right]+\left[(x_{23})_a^{\alpha_l}, (x_{23})_b^{\alpha_l}\right]+\left[(x_{33})_a^{\alpha_l}, (x_{33})_b^{\alpha_l}\right]+\left[(x_{43})_a^{\alpha_l}, (x_{43})_b^{\alpha_l}\right]\right)\leq\left[50+50\alpha_l, 250-50\alpha_l\right],$$
$$\left[(x_{ij})_a^{\alpha}, (x_{ij})_b^{\alpha}\right]\supseteq\left[(x_{ij})_a^{\alpha_{l-1}}, (x_{ij})_b^{\alpha_{l-1}}\right], \left[(x_{ij})_a^{\alpha_l}, (x_{ij})_b^{\alpha_l}\right]\geq[0,0]; i=1,2,3,4\ ;\ j=1,2,3 \text{ and } \alpha_l\in[0,1].$$

Step 2: Using the arithmetic operation,

$$\left[(p_1)_a^{\alpha}, (p_1)_b^{\alpha}\right]\left[(q_1)_a^{\alpha}, (q_1)_b^{\alpha}\right]=\begin{cases}\left[(p_1)_a^{\alpha}(q_1)_a^{\alpha}, (p_1)_b^{\alpha}(q_1)_b^{\alpha}\right]; & if\,(p_1)_a^{\alpha}\geq 0, (q_1)_a^{\alpha}\geq 0.\\ \left[(p_1)_a^{\alpha}(q_1)_b^{\alpha}, (p_1)_b^{\alpha}(q_1)_a^{\alpha}\right]; & if\,(p_1)_b^{\alpha}\leq 0, (q_1)_a^{\alpha}\geq 0.\end{cases}$$

the problem (P39) can be transformed into problem (P40).

Maximize $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$\Big[10\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right)+\left(10+10\alpha_l\right)\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right)+\left(10+12\alpha_l\right)\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right)$$
$$+\left(8+2\alpha_l\right)\left((x_{41})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right),\left(40-10\alpha_l\right)\left((x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right)+\left(40-10\alpha_l\right)\left((x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right)$$
$$+\left(34-12\alpha_l\right)\left((x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right)+\left(16-2\alpha_l\right)\left((x_{41})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right)\Big]=\Big[Z_a^{\alpha_l}+8\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}\right.$$
$$\left.+(x_{41})_a^{\alpha_l}\right)+\left(5+4\alpha_l\right)\left((x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}\right)+5\alpha\left((x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right),Z_b^{\alpha_l}+\left(16-8\alpha_l\right)\left((x_{11})_b^{\alpha_l}\right.$$
$$\left.+(x_{21})_b^{\alpha_l}+(x_{31})_b^{\alpha_l}+(x_{41})_b^{\alpha_l}\right)+\left(12-2\alpha\right)\left((x_{12})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}\right)+\left(14-5\alpha_l\right)\left((x_{13})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right)\Big],$$
$$\left[(x_{11})_a^{\alpha_l},(x_{11})_b^{\alpha_l}\right]\geq 0.40\left[(x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l},(x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right],$$
$$\left[(x_{21})_a^{\alpha_l},(x_{21})_b^{\alpha_l}\right]\geq 0.20\left[(x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l},(x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right],$$
$$\left[(x_{31})_a^{\alpha_l},(x_{31})_b^{\alpha_l}\right]\geq 0.50\left[(x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l},(x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right], \qquad \text{(P40)}$$
$$\left[(x_{12})_a^{\alpha_l},(x_{12})_b^{\alpha_l}\right]\leq 0.20\left[(x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l},(x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right],$$
$$\left[(x_{22})_a^{\alpha_l},(x_{22})_b^{\alpha_l}\right]\leq 0.40\left[(x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l},(x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right],$$
$$\left[(x_{32})_a^{\alpha_l},(x_{32})_b^{\alpha_l}\right]\leq 0.10\left[(x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l},(x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right],$$
$$\left[(x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}+(x_{41})_a^{\alpha_l},(x_{11})_b^{\alpha_l}+(x_{21})_b^{\alpha_l}+(x_{31})_b^{\alpha_l}+(x_{41})_b^{\alpha_l}\right]\leq\left[100+50\alpha_l,300-50\alpha_l\right],$$
$$\left[(x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l},(x_{12})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}\right]\leq\left[160+20\alpha_l,240-20\alpha_l\right],$$
$$\left[(x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l},(x_{13})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right]\leq\left[50+50\alpha_l,250-50\alpha_l\right],$$
$$\left[(x_{ij})_a^{\alpha_l},(x_{ij})_b^{\alpha_l}\right]\supseteq\left[(x_{ij})_a^{\alpha_{l-1}},(x_{ij})_b^{\alpha_{l-1}}\right],\left[(x_{ij})_a^{\alpha_l},(x_{ij})_b^{\alpha_l}\right]\geq[0,0];i=1,2,3,4\ ;\ j=1,2,3 \text{ and } \alpha_l\in[0,1].$$

Step 3: Using the properties $[a,b]\leq[c,d]\Rightarrow a\leq c\ \&\ b\leq d$, $[a,b]=[c,d]\Rightarrow a=c\ \&\ b=d$ and $[a,b]\subseteq[c,d]\Rightarrow a\geq c\ \&\ b\leq d$, the problem (P40) can be transformed into problem (P41).

Maximize $\left(\left[Z_a^{\alpha_l}, Z_b^{\alpha_l}\right]\right)$

Subject to

$$10\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right)+\left(10+10\alpha_l\right)\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right)+\left(10+12\alpha_l\right)\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right)$$
$$+\left(8+2\alpha_l\right)\left((x_{41})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right)=Z_a^{\alpha_l}+8\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}+(x_{41})_a^{\alpha_l}\right)+\left(5+4\alpha_l\right)\left((x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}\right.$$
$$\left.+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}\right)+5\alpha_l\left((x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right),$$

$$\left(40-10\alpha_l\right)\left((x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right)+\left(40-10\alpha_l\right)\left((x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right)+\left(34-12\alpha_l\right)\left((x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right)$$
$$+\left(16-2\alpha_l\right)\left((x_{41})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right)=Z_b^{\alpha_l}+\left(16-8\alpha_l\right)\left((x_{11})_b^{\alpha_l}+(x_{21})_b^{\alpha_l}+(x_{31})_b^{\alpha_l}+(x_{41})_b^{\alpha_l}\right)+\left(12-2\alpha_l\right)\left((x_{12})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}\right.$$
$$\left.+(x_{32})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}\right)+\left(14-5\alpha_l\right)\left((x_{13})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right),$$
$$(x_{11})_a^{\alpha_l}\geq 0.40\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right),$$
$$(x_{11})_b^{\alpha_l}\geq 0.40\left((x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right),$$
$$(x_{21})_a^{\alpha_l}\geq 0.20\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right),$$
$$(x_{21})_b^{\alpha_l}\geq 0.20\left((x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right),$$
$$(x_{31})_a^{\alpha_l}\geq 0.50\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right),$$
$$(x_{31})_b^{\alpha_l}\geq 0.50\left((x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right),$$
$$(x_{12})_a^{\alpha_l}\leq 0.20\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right), \qquad \text{(P41)}$$
$$(x_{12})_b^{\alpha_l}\leq 0.20\left((x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right),$$
$$(x_{22})_a^{\alpha_l}\leq 0.40\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right),$$
$$(x_{22})_b^{\alpha_l}\leq 0.40\left((x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right),$$
$$(x_{32})_a^{\alpha_l}\leq 0.10\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right),$$
$$(x_{32})_b^{\alpha_l}\leq 0.10\left((x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right),$$
$$(x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}+(x_{41})_a^{\alpha_l}\leq 100+50\alpha_l,$$
$$(x_{11})_b^{\alpha_l}+(x_{21})_b^{\alpha_l}+(x_{31})_b^{\alpha_l}+(x_{41})_b^{\alpha_l}\leq 300-50\alpha_l,$$
$$(x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}\leq 160+20\alpha_l,$$
$$(x_{12})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}\leq 240-20\alpha_l,$$
$$(x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\leq 50+50\alpha_l,$$
$$(x_{13})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\leq 250-50\alpha_l,$$
$$(x_{ij})_a^{\alpha_l}\leq (x_{ij})_b^{\alpha_l},(x_{ij})_a^{\alpha_l}\leq (x_{ij})_a^{\alpha_{l-1}},(x_{ij})_b^{\alpha_l}\geq (x_{ij})_b^{\alpha_{l-1}},(x_{ij})_a^{\alpha_l}\geq 0; i=1,2,3,4\ ;\ j=1,2,3 \text{ and } \alpha_l\in[0,1].$$

Step 4: Using the property $\min_{1\leq i\leq n}[a_i,b_i]=\left[\min_{1\leq i\leq n}a_i,\min_{1\leq i\leq n}b_i\right]$, problem (P41) can be transformed into problem (P42).

$$\left[\text{Maximize }\left(Z_a^{\alpha_l}\right),\text{Maximize }\left(Z_b^{\alpha_l}\right)\right]$$

Subject to

$$10\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right)+\left(10+10\alpha_l\right)\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right)+\left(10+12\alpha_l\right)\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right)$$
$$+\left(8+2\alpha_l\right)\left((x_{41})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right)=Z_a^{\alpha_l}+8\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}+(x_{41})_a^{\alpha_l}\right)+\left(5+4\alpha_l\right)\left((x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}\right.$$
$$\left.+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}\right)+5\alpha_l\left((x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right),$$
$$\left(40-10\alpha_l\right)\left((x_{11})_b^{\alpha_l}+(x_{12})_b^{\alpha_l}+(x_{13})_b^{\alpha_l}\right)+\left(40-10\alpha_l\right)\left((x_{21})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}\right)+\left(34-12\alpha_l\right)\left((x_{31})_b^{\alpha_l}+(x_{32})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}\right)$$
$$+\left(16-2\alpha_l\right)\left((x_{41})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right)=Z_b^{\alpha_l}+\left(16-8\alpha_l\right)\left((x_{11})_b^{\alpha_l}+(x_{21})_b^{\alpha_l}+(x_{31})_b^{\alpha_l}+(x_{41})_b^{\alpha_l}\right)+\left(12-2\alpha_l\right)\left((x_{12})_b^{\alpha_l}+(x_{22})_b^{\alpha_l}\right.$$
$$\left.+(x_{32})_b^{\alpha_l}+(x_{42})_b^{\alpha_l}\right)+\left(14-5\alpha_l\right)\left((x_{13})_b^{\alpha_l}+(x_{23})_b^{\alpha_l}+(x_{33})_b^{\alpha_l}+(x_{43})_b^{\alpha_l}\right),$$

$(x_{11})_a^{\alpha_l} \geq 0.40\left((x_{11})_a^{\alpha_l} + (x_{12})_a^{\alpha_l} + (x_{13})_a^{\alpha_l}\right),$

$(x_{11})_b^{\alpha_l} \geq 0.40\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right),$

$(x_{21})_a^{\alpha_l} \geq 0.20\left((x_{21})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{23})_a^{\alpha_l}\right),$

$(x_{21})_b^{\alpha_l} \geq 0.20\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right),$

$(x_{31})_a^{\alpha_l} \geq 0.50\left((x_{31})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{33})_a^{\alpha_l}\right),$

$(x_{31})_b^{\alpha_l} \geq 0.50\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right),$

$(x_{12})_a^{\alpha_l} \leq 0.20\left((x_{11})_a^{\alpha_l} + (x_{12})_a^{\alpha_l} + (x_{13})_a^{\alpha_l}\right),$ (P42)

$(x_{12})_b^{\alpha_l} \leq 0.20\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right),$

$(x_{22})_a^{\alpha_l} \leq 0.40\left((x_{21})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{23})_a^{\alpha_l}\right),$

$(x_{22})_b^{\alpha_l} \leq 0.40\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right),$

$(x_{32})_a^{\alpha_l} \leq 0.10\left((x_{31})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{33})_a^{\alpha_l}\right),$

$(x_{32})_b^{\alpha_l} \leq 0.10\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right),$

$(x_{11})_a^{\alpha_l} + (x_{21})_a^{\alpha_l} + (x_{31})_a^{\alpha_l} + (x_{41})_a^{\alpha_l} \leq 100 + 50\alpha_l,$

$(x_{11})_b^{\alpha_l} + (x_{21})_b^{\alpha_l} + (x_{31})_b^{\alpha_l} + (x_{41})_b^{\alpha_l} \leq 300 - 50\alpha_l,$

$(x_{12})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{42})_a^{\alpha_l} \leq 160 + 20\alpha_l,$

$(x_{12})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{42})_b^{\alpha_l} \leq 240 - 20\alpha_l,$

$(x_{13})_a^{\alpha_l} + (x_{23})_a^{\alpha_l} + (x_{33})_a^{\alpha_l} + (x_{43})_a^{\alpha_l} \leq 50 + 50\alpha_l,$

$(x_{13})_b^{\alpha_l} + (x_{23})_b^{\alpha_l} + (x_{33})_b^{\alpha_l} + (x_{43})_b^{\alpha_l} \leq 250 - 50\alpha_l,$

$(x_{ij})_a^{\alpha_l} \leq (x_{ij})_b^{\alpha_l}, (x_{ij})_a^{\alpha_l} \leq (x_{ij})_a^{\alpha_{l-1}}, (x_{ij})_b^{\alpha_l} \geq (x_{ij})_b^{\alpha_{l-1}}, (x_{ij})_a^{\alpha_l} \geq 0; i = 1,2,3,4 \; ; \; j = 1,2,3$ and $\alpha_l \in [0,1].$

Step 5: The problem (P42) can be transformed into the multi-objective linear programming problem (P43).

Maximize $\left(Z_a^{\alpha_l}\right)$

Maximize $\left(Z_b^{\alpha_l}\right)$

Subject to

$10\alpha_l\left((x_{11})_a^{\alpha_l} + (x_{12})_a^{\alpha_l} + (x_{13})_a^{\alpha_l}\right) + \left(10 + 10\alpha_l\right)\left((x_{21})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{23})_a^{\alpha_l}\right) + \left(10 + 12\alpha_l\right)\left((x_{31})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{33})_a^{\alpha_l}\right)$

$+\left(8 + 2\alpha_l\right)\left((x_{41})_a^{\alpha_l} + (x_{42})_a^{\alpha_l} + (x_{43})_a^{\alpha_l}\right) = Z_a^{\alpha_l} + 8\alpha_l\left((x_{11})_a^{\alpha_l} + (x_{21})_a^{\alpha_l} + (x_{31})_a^{\alpha_l} + (x_{41})_a^{\alpha_l}\right) + \left(5 + 4\alpha_l\right)\left((x_{12})_a^{\alpha_l} + (x_{22})_a^{\alpha_l}\right.$

$\left.+(x_{32})_a^{\alpha_l} + (x_{42})_a^{\alpha_l}\right) + 5\alpha_l\left((x_{13})_a^{\alpha_l} + (x_{23})_a^{\alpha_l} + (x_{33})_a^{\alpha_l} + (x_{43})_a^{\alpha_l}\right),$

$\left(40 - 10\alpha_l\right)\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right) + \left(40 - 10\alpha_l\right)\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right) + \left(34 - 12\alpha_l\right)\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right)$

$+\left(16 - 2\alpha_l\right)\left((x_{41})_b^{\alpha_l} + (x_{42})_b^{\alpha_l} + (x_{43})_b^{\alpha_l}\right) = Z_b^{\alpha_l} + \left(16 - 8\alpha_l\right)\left((x_{11})_b^{\alpha_l} + (x_{21})_b^{\alpha_l} + (x_{31})_b^{\alpha_l} + (x_{41})_b^{\alpha_l}\right) + \left(12 - 2\alpha_l\right)\left((x_{12})_b^{\alpha_l} + (x_{22})_b^{\alpha_l}\right.$

$\left.+(x_{32})_b^{\alpha_l} + (x_{42})_b^{\alpha_l}\right) + \left(14 - 5\alpha_l\right)\left((x_{13})_b^{\alpha_l} + (x_{23})_b^{\alpha_l} + (x_{33})_b^{\alpha_l} + (x_{43})_b^{\alpha_l}\right),$

$(x_{11})_a^{\alpha_l} \geq 0.40\left((x_{11})_a^{\alpha_l} + (x_{12})_a^{\alpha_l} + (x_{13})_a^{\alpha_l}\right),$

$(x_{11})_b^{\alpha_l} \geq 0.40\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right),$ (P43)

$$(x_{21})_a^{\alpha_l} \ge 0.20\left((x_{21})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{23})_a^{\alpha_l}\right),$$
$$(x_{21})_b^{\alpha_l} \ge 0.20\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right),$$
$$(x_{31})_a^{\alpha_l} \ge 0.50\left((x_{31})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{33})_a^{\alpha_l}\right),$$
$$(x_{31})_b^{\alpha_l} \ge 0.50\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right),$$
$$(x_{12})_a^{\alpha_l} \le 0.20\left((x_{11})_a^{\alpha_l} + (x_{12})_a^{\alpha_l} + (x_{13})_a^{\alpha_l}\right),$$
$$(x_{12})_b^{\alpha_l} \le 0.20\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right),$$
$$(x_{22})_a^{\alpha_l} \le 0.40\left((x_{21})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{23})_a^{\alpha_l}\right),$$
$$(x_{22})_b^{\alpha_l} \le 0.40\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right),$$
$$(x_{32})_a^{\alpha_l} \le 0.10\left((x_{31})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{33})_a^{\alpha_l}\right),$$
$$(x_{32})_b^{\alpha_l} \le 0.10\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right),$$
$$(x_{11})_a^{\alpha_l} + (x_{21})_a^{\alpha_l} + (x_{31})_a^{\alpha_l} + (x_{41})_a^{\alpha_l} \le 100 + 50\alpha_l,$$
$$(x_{11})_b^{\alpha_l} + (x_{21})_b^{\alpha_l} + (x_{31})_b^{\alpha_l} + (x_{41})_b^{\alpha_l} \le 300 - 50\alpha_l,$$
$$(x_{12})_a^{\alpha_l} + (x_{22})_a^{\alpha_l} + (x_{32})_a^{\alpha_l} + (x_{42})_a^{\alpha_l} \le 160 + 20\alpha_l,$$
$$(x_{12})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{42})_b^{\alpha_l} \le 240 - 20\alpha_l,$$
$$(x_{13})_a^{\alpha_l} + (x_{23})_a^{\alpha_l} + (x_{33})_a^{\alpha_l} + (x_{43})_a^{\alpha_l} \le 50 + 50\alpha_l,$$
$$(x_{13})_b^{\alpha_l} + (x_{23})_b^{\alpha_l} + (x_{33})_b^{\alpha_l} + (x_{43})_b^{\alpha_l} \le 250 - 50\alpha_l,$$
$$(x_{ij})_a^{\alpha_l} \le (x_{ij})_b^{\alpha_l}, (x_{ij})_a^{\alpha_l} \le (x_{ij})_a^{\alpha_{l-1}}, (x_{ij})_b^{\alpha_l} \ge (x_{ij})_b^{\alpha_{l-1}}, (x_{ij})_a^{\alpha_l} \ge 0; i = 1,2,3,4\ ;\ j = 1,2,3 \text{ and } \alpha_l \in [0,1].$$

Step 6: The problem (P43), can be transformed into two problems (P44) and (P45).

Maximize $\left(Z_b^{\alpha_l}\right)$

Subject to

$$\left(40-10\alpha_l\right)\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right) + \left(40-10\alpha_l\right)\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right) + \left(34-12\alpha_l\right)\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right)$$
$$+\left(16-2\alpha_l\right)\left((x_{41})_b^{\alpha_l} + (x_{42})_b^{\alpha_l} + (x_{43})_b^{\alpha_l}\right) = Z_b^{\alpha_l} + \left(16-8\alpha_l\right)\left((x_{11})_b^{\alpha_l} + (x_{21})_b^{\alpha_l} + (x_{31})_b^{\alpha_l} + (x_{41})_b^{\alpha_l}\right) + \left(12-2\alpha_l\right)\left((x_{12})_b^{\alpha_l} + (x_{22})_b^{\alpha_l}\right.$$
$$\left.+(x_{32})_b^{\alpha_l} + (x_{42})_b^{\alpha_l}\right) + \left(14-5\alpha_l\right)\left((x_{13})_b^{\alpha_l} + (x_{23})_b^{\alpha_l} + (x_{33})_b^{\alpha_l} + (x_{43})_b^{\alpha_l}\right),$$
$$(x_{11})_b^{\alpha_l} \ge 0.40\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right),$$
$$(x_{21})_b^{\alpha_l} \ge 0.20\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right),$$
$$(x_{31})_b^{\alpha_l} \ge 0.50\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right), \qquad \text{(P44)}$$
$$(x_{12})_b^{\alpha_l} \le 0.20\left((x_{11})_b^{\alpha_l} + (x_{12})_b^{\alpha_l} + (x_{13})_b^{\alpha_l}\right),$$
$$(x_{22})_b^{\alpha_l} \le 0.40\left((x_{21})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{23})_b^{\alpha_l}\right),$$
$$(x_{32})_b^{\alpha_l} \le 0.10\left((x_{31})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{33})_b^{\alpha_l}\right),$$
$$(x_{11})_b^{\alpha_l} + (x_{21})_b^{\alpha_l} + (x_{31})_b^{\alpha_l} + (x_{41})_b^{\alpha_l} \le 300 - 50\alpha_l,$$
$$(x_{12})_b^{\alpha_l} + (x_{22})_b^{\alpha_l} + (x_{32})_b^{\alpha_l} + (x_{42})_b^{\alpha_l} \le 240 - 20\alpha_l,$$
$$(x_{13})_b^{\alpha_l} + (x_{23})_b^{\alpha_l} + (x_{33})_b^{\alpha_l} + (x_{43})_b^{\alpha_l} \le 250 - 50\alpha_l,$$
$$(x_{ij})_b^{\alpha_l} \ge (x_{ij})_b^{\alpha_{l-1}}, (x_{ij})_b^{\alpha_l} \ge 0; i = 1,2,3,4\ ;\ j = 1,2,3 \text{ and } \alpha_l \in [0,1].$$

Maximize $\left(Z_a^{\alpha_l}\right)$

Subject to

$$10\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right)+\left(10+10\alpha_l\right)\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right)+\left(10+12\alpha_l\right)\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right)$$
$$+\left(8+2\alpha_l\right)\left((x_{41})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right)=Z_a^{\alpha_l}+8\alpha_l\left((x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}+(x_{41})_a^{\alpha_l}\right)+\left(5+4\alpha_l\right)\left((x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}\right.$$
$$\left.+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}\right)+5\alpha_l\left((x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\right),$$
$$(x_{11})_a^{\alpha_l}\geq 0.40\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right),$$
$$(x_{21})_a^{\alpha_l}\geq 0.20\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right),$$
$$(x_{31})_a^{\alpha_l}\geq 0.50\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right),$$
$$(x_{12})_a^{\alpha_l}\leq 0.20\left((x_{11})_a^{\alpha_l}+(x_{12})_a^{\alpha_l}+(x_{13})_a^{\alpha_l}\right), \qquad \text{(P45)}$$
$$(x_{22})_a^{\alpha_l}\leq 0.40\left((x_{21})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}\right),$$
$$(x_{32})_a^{\alpha_l}\leq 0.10\left((x_{31})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}\right),$$
$$(x_{11})_a^{\alpha_l}+(x_{21})_a^{\alpha_l}+(x_{31})_a^{\alpha_l}+(x_{41})_a^{\alpha_l}\leq 100+50\alpha_l,$$
$$(x_{12})_a^{\alpha_l}+(x_{22})_a^{\alpha_l}+(x_{32})_a^{\alpha_l}+(x_{42})_a^{\alpha_l}\leq 160+20\alpha_l,$$
$$(x_{13})_a^{\alpha_l}+(x_{23})_a^{\alpha_l}+(x_{33})_a^{\alpha_l}+(x_{43})_a^{\alpha_l}\leq 50+50\alpha_l,$$
$$(x_{ij})_a^{\alpha_l}\leq (x_{ij})_a^{\alpha_{l-1}},\ (x_{ij})_a^{\alpha_l}\geq 0; i=1,2,3,4\ ;\ j=1,2,3.$$

Step 7: The optimal solution $\left\{(x_{ij})_b^{\alpha_l}\right\}$ of problem (P44) without considering the constraint $(x_{ij})_b^{\alpha_l}\geq (x_{ij})_b^{\alpha_{l-1}}; i=1,2,3,4; j=1,2,3$ and the optimal solution $\left\{(x_{ij})_a^{\alpha_l}\right\}$ of problem (P45) without considering the constraint $(x_{ij})_a^{\alpha_l}\leq (x_{ij})_a^{\alpha_{l-1}}; i=1,2,3,4; j=1,2,3$ for $\alpha_l=\alpha_1=1$ is shown in **Table 4**.

$\alpha's$	Optimal Solution
1	$(x_{11opt})_b^1=96,(x_{12opt})_b^1=48,(x_{13opt})_b^1=96,(x_{21opt})_b^1=154,(x_{22opt})_b^1=172,(x_{23opt})_b^1=104$ $(x_{31opt})_b^1=0,(x_{32opt})_b^1=0,(x_{33opt})_b^1=0,(x_{41opt})_b^1=0,(x_{42opt})_b^1=0,(x_{43opt})_b^1=0,$ $\left(Z_b^1\right)_{opt}=14100.$
	$(x_{11opt})_a^1=0,(x_{12opt})_a^1=0,(x_{13opt})_a^1=0,(x_{21opt})_a^1=150,(x_{22opt})_a^1=166.67,(x_{23opt})_a^1=100$ $(x_{31opt})_a^1=0,(x_{32opt})_a^1=0,(x_{33opt})_a^1=0,(x_{41opt})_a^1=0,(x_{42opt})_a^1=0,(x_{43opt})_a^1=0,$ $\left(Z_a^1\right)_{opt}=5133.33.$

Table 4. Solutions of problem (P44) and problem (P45) at $\alpha_l=\alpha_1=1$

Step 8: The optimal solution $\left\{(x_{ij})_b^{\alpha_l}\right\}$ of problem (P44) and the optimal solution $\left\{(x_{ij})_a^{\alpha_l}\right\}$ of problem (P45) for $\alpha_l=\alpha_2=0.9,\ \alpha_l=\alpha_3=0.8,\ ...,\alpha_l=\alpha_{11}=0$ is shown in **Table 5**.

$\alpha's$		Optimal Solution

0.9		$(x_{11opt})_b^{0.9}=96, (x_{12opt})_b^{0.9}=48, (x_{13opt})_b^{0.9}=96, (x_{21opt})_b^{0.9}=159,$ $(x_{31opt})_b^{0.9}=0, (x_{32opt})_b^{0.9}=0, (x_{33opt})_b^{0.9}=0, (x_{41opt})_b^{0.9}=0, (x_{42opt})$ $\left(Z_b^{0.9}\right)_{opt}=14686.$
		$(x_{11opt})_a^{0.9}=0, (x_{12opt})_a^{0.9}=0, (x_{13opt})_a^{0.9}=0, (x_{21opt})_a^{0.9}=145,$ $(x_{31opt})_a^{0.9}=0, (x_{32opt})_a^{0.9}=0, (x_{33opt})_a^{0.9}=0, (x_{41opt})_a^{0.9}=0, (x_{42opt})$ $\left(Z_a^{0.9}\right)_{opt}=4752.5.$
0.8		$(x_{11opt})_b^{0.8}=96, (x_{12opt})_b^{0.8}=48, (x_{13opt})_b^{0.8}=96, (x_{21opt})_b^{0.8}=164,$ $(x_{31opt})_b^{0.8}=0, (x_{32opt})_b^{0.8}=0, (x_{33opt})_b^{0.8}=0, (x_{41opt})_b^{0.8}=0, (x_{42opt})$ $\left(Z_b^{0.8}\right)_{opt}=15282.40.$
		$(x_{11opt})_a^{0.8}=0, (x_{12opt})_a^{0.8}=0, (x_{13opt})_a^{0.8}=0, (x_{21opt})_a^{0.8}=140,$ $(x_{31opt})_a^{0.8}=0, (x_{32opt})_a^{0.8}=0, (x_{33opt})_a^{0.8}=0, (x_{41opt})_a^{0.8}=0, (x_{42opt})$ $\left(Z_a^{0.8}\right)_{opt}=4386.67.$
0.7		$(x_{11opt})_b^{0.7}=96, (x_{12opt})_b^{0.7}=48, (x_{13opt})_b^{0.7}=96, (x_{21opt})_b^{0.7}=169,$ $(x_{31opt})_b^{0.7}=0, (x_{32opt})_b^{0.7}=0, (x_{33opt})_b^{0.7}=0, (x_{41opt})_b^{0.7}=0, (x_{42opt})$ $\left(Z_b^{0.7}\right)_{opt}=15888.9.$
		$(x_{11opt})_a^{0.7}=0, (x_{12opt})_a^{0.7}=0, (x_{13opt})_a^{0.7}=0, (x_{21opt})_a^{0.7}=135,$ $(x_{31opt})_a^{0.7}=0, (x_{32opt})_a^{0.7}=0, (x_{33opt})_a^{0.7}=0, (x_{41opt})_a^{0.7}=0, (x_{42opt})$ $\left(Z_a^{0.7}\right)_{opt}=4035.83.$
0.6		$(x_{11opt})_b^{0.6}=96, (x_{12opt})_b^{0.6}=48, (x_{13opt})_b^{0.6}=96, (x_{21opt})_b^{0.6}=174,$ $(x_{31opt})_b^{0.6}=0, (x_{32opt})_b^{0.6}=0, (x_{33opt})_b^{0.6}=0, (x_{41opt})_b^{0.6}=0, (x_{42opt})$ $\left(Z_b^{0.6}\right)_{opt}=16505.6.$
		$(x_{11opt})_a^{0.6}=0, (x_{12opt})_a^{0.6}=0, (x_{13opt})_a^{0.6}=0, (x_{21opt})_a^{0.6}=130,$ $(x_{31opt})_a^{0.6}=0, (x_{32opt})_a^{0.6}=0, (x_{33opt})_a^{0.6}=0, (x_{41opt})_a^{0.6}=0, (x_{42opt})$ $\left(Z_a^{0.6}\right)_{opt}=3700.$
0.5		$(x_{11opt})_b^{0.5}=96, (x_{12opt})_b^{0.5}=48, (x_{13opt})_b^{0.5}=96, (x_{21opt})_b^{0.5}=179,$ $(x_{31opt})_b^{0.5}=0, (x_{32opt})_b^{0.5}=0, (x_{33opt})_b^{0.5}=0, (x_{41opt})_b^{0.5}=0, (x_{42opt})$ $\left(Z_b^{0.5}\right)_{opt}=17132.5.$
		$(x_{11opt})_a^{0.5}=0, (x_{12opt})_a^{0.5}=0, (x_{13opt})_a^{0.5}=0, (x_{21opt})_a^{0.5}=125,$ $(x_{31opt})_a^{0.5}=0, (x_{32opt})_a^{0.5}=0, (x_{33opt})_a^{0.5}=0, (x_{41opt})_a^{0.5}=0, (x_{42opt})$ $\left(Z_a^{0.5}\right)_{opt}=3379.17.$

0.4		$(x_{11opt})_b^{0.4}=96,(x_{12opt})_b^{0.4}=48,(x_{13opt})_b^{0.4}=96,(x_{21opt})_b^{0.4}=184,($ $(x_{31opt})_b^{0.4}=0,(x_{32opt})_b^{0.4}=0,(x_{33opt})_b^{0.4}=0,(x_{41opt})_b^{0.4}=0,(x_{42opt})$ $\left(Z_b^{0.4}\right)_{opt}=17769.6.$
		$(x_{11opt})_a^{0.4}=0,(x_{12opt})_a^{0.4}=0,(x_{13opt})_a^{0.4}=0,(x_{21opt})_a^{0.4}=120,(x_{22op}$ $(x_{31opt})_a^{0.4}=0,(x_{32opt})_a^{0.4}=0,(x_{33opt})_a^{0.4}=0,(x_{41opt})_a^{0.4}=0,(x_{42opt})$ $\left(Z_a^{0.4}\right)_{opt}=3073.33.$
0.3		$(x_{11opt})_b^{0.3}=96,(x_{12opt})_b^{0.3}=48,(x_{13opt})_b^{0.3}=96,(x_{21opt})_b^{0.3}=189,($ $(x_{31opt})_b^{0.3}=0,(x_{32opt})_b^{0.3}=0,(x_{33opt})_b^{0.3}=0,(x_{41opt})_b^{0.3}=0,(x_{42opt})$ $\left(Z_b^{0.3}\right)_{opt}=18415.9.$
		$(x_{11opt})_a^{0.3}=2\times10^{-9},(x_{12opt})_a^{0.3}=0,(x_{13opt})_a^{0.3}=0,(x_{21opt})_a^{0.3}=115$ $(x_{31opt})_a^{0.3}=0,(x_{32opt})_a^{0.3}=0,(x_{33opt})_a^{0.3}=0,(x_{41opt})_a^{0.3}=0,(x_{42opt})$ $\left(Z_a^{0.3}\right)_{opt}=2782.5.$
0.2		$(x_{11opt})_b^{0.2}=96,(x_{12opt})_b^{0.2}=48,(x_{13opt})_b^{0.2}=96,(x_{21opt})_b^{0.2}=194,($ $(x_{31opt})_b^{0.2}=0,(x_{32opt})_b^{0.2}=0,(x_{33opt})_b^{0.2}=0,(x_{41opt})_b^{0.2}=0,(x_{42opt})$ $\left(Z_b^{0.2}\right)_{opt}=19074.4.$
		$(x_{11opt})_a^{0.2}=0,(x_{12opt})_a^{0.2}=0,(x_{13opt})_a^{0.2}=0,(x_{21opt})_a^{0.2}=110,(x_{22op}$ $(x_{31opt})_a^{0.2}=0,(x_{32opt})_a^{0.2}=0,(x_{33opt})_a^{0.2}=0,(x_{41opt})_a^{0.2}=0,(x_{42opt})$ $\left(Z_a^{0.2}\right)_{opt}=2506.67.$
0.1		$(x_{11opt})_b^{0.1}=96,(x_{12opt})_b^{0.1}=48,(x_{13opt})_b^{0.1}=96,(x_{21opt})_b^{0.1}=199,($ $(x_{31opt})_b^{0.1}=0,(x_{32opt})_b^{0.1}=0,(x_{33opt})_b^{0.1}=0,(x_{41opt})_b^{0.1}=0,(x_{42opt})$ $\left(Z_b^{0.1}\right)_{opt}=19742.1.$
		$(x_{11opt})_a^{0.1}=0,(x_{12opt})_a^{0.1}=0,(x_{13opt})_a^{0.1}=0,(x_{21opt})_a^{0.1}=105,(x_{22opt}$ $(x_{31opt})_a^{0.1}=0,(x_{32opt})_a^{0.1}=0,(x_{33opt})_a^{0.1}=0,(x_{41opt})_a^{0.1}=0,(x_{42opt})$ $\left(Z_a^{0.1}\right)_{opt}=2245.83.$
0		$(x_{11opt})_b^{0}=96,(x_{12opt})_b^{0}=48,(x_{13opt})_b^{0}=96,(x_{21opt})_b^{0}=204,(x_{22opt}$ $(x_{31opt})0=0,(x_{32opt})_b^{0}=0,(x_{33opt})_b^{0}=0,(x_{41opt})_b^{0}=0,(x_{42opt})_b^{0}=0$ $\left(Z_b^{0}\right)_{opt}=20420.$
		$(x_{11opt})_a^{0}=0,(x_{12opt})_a^{0}=0,(x_{13opt})_a^{0}=0,(x_{21opt})_a^{0}=100,(x_{22opt})_a^{0}=$ $(x_{31opt})_a^{0}=0,(x_{32opt})_a^{0}=0,(x_{33opt})_a^{0}=0,(x_{41opt})_a^{0}=0,(x_{42opt})_a^{0}=0$ $\left(Z_a^{0}\right)_{opt}=2000.$

Table 5. Solutions of problem (P44) and problem (P45) at $\alpha_l=\alpha_2=0.9,\ \alpha_l=\alpha_3=0.8,\ \ldots,\alpha_l=\alpha_{11}=0$

Step 9: Using the optimal solution of problem (P39), obtained in Step 7 and Step 8 under different α -cut levels, and regression analysis, the obtained membership function for the fuzzy optimal value $(\tilde{Z}_0)$ and fuzzy optimal solution $(\tilde{x}_{ij})$ of problem (P38) are as follows:

Membership function for fuzzy optimal value $(\tilde{Z}_0)$ is

$$\mu_{\tilde{Z}_0}(x)=\begin{cases}0.0003x-0.5976 & ;\ 2000\le x<5133.33\\ 1 & ;\ 5133.33\le x\le 14100\\ -0.0002x+3.2176 & ;\ 14100<x\le 20420\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{11})$ is

$$\mu_{\tilde{x}_{11}}(x)=\begin{cases}1 & ;\ 0\le x\le 96\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{12})$ is

$$\mu_{\tilde{x}_{12}}(x)=\begin{cases}1 & ;\ 0\le x\le 48\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{13})$ is

$$\mu_{\tilde{x}_{13}}(x)=\begin{cases}1 & ;\ 0\le x\le 96\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{21})$ is

$$\mu_{\tilde{x}_{21}}(x)=\begin{cases}\dfrac{x-100}{50} & ;\ 100\le x<150\\ 1 & ;\ 150\le x\le 154\\ \dfrac{204-x}{50} & ;\ 154<x\le 204\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{22})$ is

$$\mu_{\tilde{x}_{22}}(x)=\begin{cases}\dfrac{x-100}{66.67} & ;\ 100\le x<166.67\\ 1 & ;\ 166.67\le x\le 172\\ \dfrac{192-x}{20} & ;\ 172<x\le 192\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{23})$ is

$$\mu_{\tilde{x}_{23}}(x)=\begin{cases}\frac{x-50}{50} & ;\ 50\le x<100\\ 1 & ;\ 100\le x\le 104\\ \frac{154-x}{50} & ;\ 104<x\le 154\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for optimal fuzzy decision variable $(\tilde{x}_{31})$ is

$$\mu_{\tilde{x}_{31}}(x)=\begin{cases}1 & ;\ x=0\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for fuzzy optimal decision variable $(\tilde{x}_{32})$ is

$$\mu_{\tilde{x}_{32}}(x)=\begin{cases}1 & ;\ x=0\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for fuzzy optimal decision variable $(\tilde{x}_{33})$ is

$$\mu_{\tilde{x}_{33}}(x)=\begin{cases}1 & ;\ x=0\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for fuzzy optimal decision variable $(\tilde{x}_{41})$ is

$$\mu_{\tilde{x}_{41}}(x)=\begin{cases}1 & ;\ x=0\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for fuzzy optimal decision variable $(\tilde{x}_{42})$ is

$$\mu_{\tilde{x}_{42}}(x)=\begin{cases}1 & ;\ x=0\\ 0 & ;\ \text{otherwise}\end{cases}$$

Membership function for fuzzy optimal decision variable $(\tilde{x}_{43})$ is

$$\mu_{\tilde{x}_{43}}(x)=\begin{cases}1 & ;\ x=0\\ 0 & ;\ \text{otherwise}\end{cases}$$

6. Conclusion

On the basis of the present study, it can be concluded that there are some errors in the existing method[3]. Also, some assumptions are considered in the existing methods[2,5,7,8]. However, in the proposed Mehar method, all the errors of the existing method[3] are resolved as well as the assumptions, considered in the existing methods[2,5,7,8], are not considered. Hence, it is better to use the proposed Mehar method instead of the existing methods[2,3,5,7,8].

Appendix A

Definition A1 [4]: The characteristic function μ_A of a crisp set $A\subseteq X$ assigns a value, either 0 or 1, to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}}: X\to[0,1]$. The assigned value indicates the membership grade of the element in the set A.

The function $\mu_{\tilde{A}}$ is called the membership function, and the set $\tilde{A}=\{(x,\mu_{\tilde{A}}(x)); x\in X\}$ defined by $\mu_{\tilde{A}}(x)$ for each $x\in X$, is called a fuzzy set.

Definition A2 [3]: The α -level set of a fuzzy set $\tilde{A}$ is defined as an ordinary set $[\tilde{A}]_\alpha$ for which the degree of its membership function exceeds the level α :

$$[\tilde{A}]_\alpha = \{x \mid \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1]\}$$

Definition A3 [3]: Let $F_0(\mathbb{R})$ denote a set of all fuzzy numbers on $\mathbb{R}$. For any $\tilde{a} \in F_0(\mathbb{R})$, suppose two α -levels (α, β) are selected to cut $\tilde{a}$. Thus two α -level sets can be formulated as:

$$\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right] \text{ and } \left[(q_1)_a^\beta, (q_1)_b^\beta\right]$$

if $\alpha \geq \beta$, then

$$\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right] \subseteq \left[(q_1)_a^\beta, (q_1)_b^\beta\right], \text{ namely } (p_1)_a^\alpha \geq (q_1)_a^\beta, (p_1)_b^\alpha \leq (q_1)_b^\beta .$$

Definition A4 [5]: A fuzzy number $\tilde{A}$ is said to be non-negative fuzzy number if the domain of its membership function is set of non-negative real numbers $(\mathbb{R}^+)$ i.e. $\mu_{\tilde{A}} : \mathbb{R}^+ \rightarrow [0,1]$. The set of non-negative fuzzy numbers may be represented by $F(\mathbb{R}^+)$.

Definition A5 [4]: A fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{(x-a)}{b-a} & ; a \leq x < b \\ 1 & ; b \leq x \leq c \\ \dfrac{(d-x)}{d-c} & ; c < x \leq d \\ 0 & ; \text{otherwise} \end{cases}$$

Definition A6 [5]: A trapezoidal fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be unrestricted trapezoidal fuzzy number if and only if $a,b,c,d \in \mathbb{R}$.

Definition A7 [4]: A trapezoidal fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be non-negative trapezoidal fuzzy number if and only if $a \geq 0$.

Definition A8 [4]: Two trapezoidal fuzzy numbers $\tilde{A} = (a_1,b_1,c_1,d_1)$ and $\tilde{B} = (a_2,b_2,c_2,d_2)$ are said to be equal if and only if $a_1 = a_2, b_1 = b_2, c_1 = c_2$ and $d_1 = d_2$.

Arithmetic Operations A9 [3]:

For the intervals $\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right]$ and $\left[(q_1)_a^\alpha, (q_1)_b^\alpha\right]$, we can define

(i) $\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right] + \left[(q_1)_a^\alpha, (q_1)_b^\alpha\right] = \left[(p_1)_a^\alpha + (q_1)_a^\alpha, (p_1)_b^\alpha + (q_1)_b^\alpha\right].$

(ii) $\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right] - \left[(q_1)_a^\alpha, (q_1)_b^\alpha\right] = \left[(p_1)_a^\alpha - (q_1)_b^\alpha, (p_1)_b^\alpha - (q_1)_a^\alpha\right].$

(iii) $\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right]\left[(q_1)_a^\alpha, (q_1)_b^\alpha\right] = \begin{bmatrix} (p_1)_a^\alpha (q_1)_a^\alpha \wedge (p_1)_a^\alpha (q_1)_b^\alpha \wedge (p_1)_b^\alpha (q_1)_a^\alpha \wedge (p_1)_b^\alpha (q_1)_b^\alpha, \\ (p_1)_a^\alpha (q_1)_a^\alpha \vee (p_1)_a^\alpha (q_1)_b^\alpha \vee (p_1)_b^\alpha (q_1)_a^\alpha \vee (p_1)_b^\alpha (q_1)_b^\alpha \end{bmatrix}.$

(iv)The order relation "$\leq$" and "=" are defined by:

$$\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right] \leq \left[(q_1)_a^\alpha, (q_1)_b^\alpha\right] \text{ if and only if } (p_1)_a^\alpha \leq (q_1)_a^\alpha, (p_1)_b^\alpha \leq (q_1)_b^\alpha .$$

$$\left[(p_1)_a^\alpha, (p_1)_b^\alpha\right] = \left[(q_1)_a^\alpha, (q_1)_b^\alpha\right] \text{ if and only if } (p_1)_a^\alpha = (q_1)_a^\alpha, (p_1)_b^\alpha = (q_1)_b^\alpha .$$

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