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Column-Wise Relative Degree and it's Properties

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ABSTRACT

Many questions of control theory are well studied for systems which satisfy to the relative degree definition. If this definition is fulfilled then there exists linear state-space transform reducing system to a very convenient canonical form where zero dynamics is a part of system's equations. Algorithms of such reduction are well-known. However, there exist systems which don't satisfy this definition. Such systems are the subject of investigation in the presented paper. To investigate their properties here we suggest to consider an analogue of the classical relative degree definition – the so-called column-wise relative degree. It turned out that this definition is satisfied in some cases when classical relative degree doesn't exist. We introduce this notion here, investigate it properties and suggest algorithm for reducing systems to the column-wise relative degree compliant form if possible. It is possible to show that systems with column-wise relative degree also can be reduced to a convenient canonical form by a linear state-space transformation. Some problems arise from the fact that some systems which do not have relative degree can be reduced to a form with it using linear inputs or outputs transform. Here we show that this is an interesting mathematical problem, which can be solved with the help of properties of relative degree, formulated and proved in this paper.

***Keywords*:** Control Theory; Linear Dynamical Systems; Relative Degree; Zero Dynamics; Canonical Forms

1. Introduction

The notion of zero dynamics (i.e. the dynamics of the system with respect to zero output) plays an important role in control theory. To analyze zero dynamics (i.e. find its specter, dimension and so on) the notion of system's relative degree is usually used. For linear time-invariant SISO-systems the above problems are solved completely. However, for the vector (MIMO) systems there exist unsolved problems (even for linear MIMO systems).

In the present paper the following discrete-time linear time-invariant system is considered:

$$\begin{cases} x^{t+1} = Ax^t + Bu^t \\ y^t = Cx^t, \qquad t = 0,1,2,\dots \end{cases} \qquad (1)$$

where state-space vector $x^t \in \mathrm{R}^n$, input and output $u^t, y^t \in \mathrm{R}^l$, (i.e. the system is "square"), A, B and C – are constant matrices of appropriate sizes.

Let us recall the classical notion of relative degree (RD).

Definition 1. A vector $r = (r_1, \dots, r_l)$ - is called the relative degree vector for system (1) if the following conditions are satisfied:

1. $C_i A^{j-1} B = 0, j = 1, \dots, r_i - 1;\ C_i A^{r_i - 1} B \neq 0$, for each $i = 1, \dots, l$.
2. $det H(r) \neq 0$, where $H(r) = \begin{pmatrix} C_1 A^{r_1-1} B \\ \cdots \\ C_l A^{r_l-1} B \end{pmatrix}$.

Here C_i denotes i-th row of matrix C, that's why whenever the conditions of Definition 1 are satisfied for vector r, we call such a vector "row-wise" relative degree. Whenever the conditions of Definition 1 are satisfied, there exists a nonsingular coordinate change such that the transformed system has the following form:

$$\begin{cases} (\tilde{x})^{t+1} = A_{11}(\tilde{x})^t + A_{12}(y)^t, \qquad \tilde{x} \in \mathrm{R}^{n-|r|} \\ (\tilde{y}^i_j)^{t+1} = (\tilde{y}^i_{j+1})^t, \qquad j = 1, \dots, r_i - 1, i = 1, \dots, l, \tilde{y}^i_j \in \mathrm{R}^1; \\ \begin{pmatrix} \tilde{y}^1_{r_1} \\ \vdots \\ \tilde{y}^l_{r_l} \end{pmatrix}^{t+1} = A_{21}(\tilde{x})^t + A_{22}(\tilde{y})^t + H(r)(u)^t, \end{cases}$$

where $y_i = \widetilde{y}_1^i, i = 1, \ldots, l;\ |r| = r_1 + \ldots + r_l;\ \widetilde{y} = (\widetilde{y}_1^1, \ldots, \widetilde{y}_l$ — denotes new system's phase vector. Note that the following part of phase vector $(\widetilde{y}^i)^t = ((\widetilde{y}_1^i)^t, \ldots, (\widetilde{y}_{r_i}^i)^t)^T$ is system's output y^i at time $t, t+1, \ldots, t+r_i-1$. Whenever $y^t \equiv 0$ for all $t \geq t^*$, the condition $\widetilde{y}^t \equiv 0$ holds since some moment t, and system's dynamics is described by the first subsystem, namely, by the equation $\widetilde{x}^{t+1} = A_{11}\widetilde{x}^t$.

The dimension of zero dynamics is equal to $n - |r|$, and it is uniquely identified by system's relative degree vector. The system can be transformed to the form (2) iff it has vector relative degree.

The conditions of Definition 1 can be inconsistent for a controllable and observable system. For example, the following system

$$\begin{cases} x_1^{t+1} = u_1^t \\ x_2^{t+1} = x_3^t \\ x_3^{t+1} = u_2^t \end{cases} \qquad \begin{cases} y_1^t = x_1^t \\ y_2^t = x_1^t + x_2^t \end{cases}$$

does not have relative degree. Indeed, vector r = (1,1) satisfies condition 1 of Definition 1, but the matrix

$$H(1,1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is singular, so the second condition is not satisfied.

It is easy to see that Definition 1 is invariant under nonsingular coordinate change. However, the definition is not invariant under nonsingular output change $\widetilde{y} = Ty$, where $T \in \mathrm{R}^{l \times l}$ is a nonsingular matrix. Namely, in the above example output change $\widetilde{y}_1^t = y_1^t, \widetilde{y}_2^t = y_2^t - y_1^t$, (i.e. $\widetilde{y}_1^t = x_1^t, \widetilde{y}_2^t = x_2^t$) leads to the system with output $\widetilde{y}$ and this system has relative degree vector $r = (1,2)$.

Then the following questions arise: when does an output transformation reducing the system to a form with relative degree exist, and how to find such transformation?

These questions are crucial part of the solution of some control problems -- for example, inverse problem for linear time-invariant dynamical system. Note that in the literature (namely, in[2, p.72-82]) examples of systems reducible to a form with relative degree, as well as non-reducible ones are given.

Namely, in the paper[3] it was proved that every controllable and observable third-order system with two inputs and two outputs can be reduced using linear nonsingular output change to a form with relative degree. It was also proved that in general case this condition does not hold, i.e. there exist systems that cannot be reduced to a form with relative degree using such output change. Let us also give an example of latter system.

$$\begin{cases} x_1^{t+1} = x_2^t + x_4^t + u_1^t \\ x_2^{t+1} = x_1^t + x_2^t + x_4^t \\ x_3^{t+1} = x_2^t + x_4^t + u_2^t \\ x_4^{t+1} = x_2^t + x_3^t + x_4^t \end{cases} \qquad \begin{cases} y_1^t = x_3^t \\ y_2^t = x_4^t \end{cases} \qquad (3)$$

As it was shown in paper[4], this system does not have a classical relative degree and cannot be reduced to a form with relative degree using nonsingular linear output change.

Indeed, for this system

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (4)$$

It is easy to verify, that system (4) is observable and controllable. But for every linear output transform $T \in \mathrm{R}^{2\times 2}, |T| \neq 0$ we receive:

$$\widetilde{C} = TC = \begin{pmatrix} 0 & 0 & c_{13} & c_{14} \\ 0 & 0 & c_{23} & c_{24} \end{pmatrix}, \begin{vmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{vmatrix} \neq 0$$

If $c_{13} = 0$, then $\widetilde{C}_1 AB = [0\ \ c_{14}]$, and $c_{14} \neq 0$, otherwise first row of outputs matrix $\widetilde{C}$ would be zero. Also $c_{23} \neq 0$, otherwise rank of outputs matrix would be incomplete. But in this case

$$|H(2,1)| = \begin{vmatrix} 0 & c_{14} \\ 0 & c_{23} \end{vmatrix} = 0.$$

If $c_{13} \neq 0$, then for existance of RD we must demand $c_{23} = 0$. Then $\widetilde{C}_2 AB = [0\ \ c_{24}]$, where $c_{24} \neq 0$, otherwise the second row of outputs matrix would be zero. But in this case

$$|H(1,2)| = \begin{vmatrix} 0 & c_{13} \\ 0 & c_{24} \end{vmatrix} = 0.$$

Thus, for the specified system it is impossible to find linear transform reducing it to a form with RD. This example shows that such transform doesn't always exist.

However, to reduce system (1) to a convenient canonical form it is possible to suggest an alternative method, namely, to consider an analogue of RD by changing rows of C to columns of B — the so-called column-wise RD. We shall discuss this notion in the presented paper.

2. Column-wise relative degree and it's properties

Let's introduce column-wise RD definition.

Definition 2. Vector $r=(r_1,\dots,r_l)$ - is a vector of column-wise relative degree for system (1), if the following conditions are fulfilled:

1) $CA^{j-1}B_i=0,\ j=1,\dots,r_i-1;\ CA^{r_i-1}B_i\neq 0$, for all $i=1,\dots,l$.

2) $detH(r_1,\dots,r_l)=det(CA^{r_1-1}B_1\,|\dots|\,CA^{r_l-1}B_l)\neq 0$,

where B_i are columns of matrix B.

It is necessary to note, that conditions 1) and 2) of Definition 2 may be inconsistents, and, thus, for some systems this definition is not satisfied.

RD defined by Definition 1 may be called row-wise. Let's note, that if $r=(r_1,\dots,r_l)$ — column-wise RD for system (1), then this vector appears to be row-wise RD for conjugated system

$$\begin{cases} x^{t+1}=A^Tx^t+C^Tu^t \\ y^t=B^Tx^t, \qquad t=0,1,2,\dots . \end{cases}$$

It is an actual question whether reducibility of system (1) to a form with row-wise RD is equivalent to it's reducibility to a form with column-wise RD ? The answer for this question is negative. Let's turn back to the example (3-4), considered earlier. In it system is unreducable to a form with row-wise RD, but it's column-wise RD exists and is described by the following considerations:

$$CB_1=\begin{pmatrix}0\\0\end{pmatrix},\ CAB_1=\begin{pmatrix}0\\0\end{pmatrix},\ CA^2B_1=\begin{pmatrix}1\\1\end{pmatrix}\ \Rightarrow\ r_1=3$$

$$CB_2=\begin{pmatrix}1\\0\end{pmatrix}\ \Rightarrow\ r_2=1$$

$$H(3,1)=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

In case of reducibility of system (1) to a form with column-wise RD, system (1) itself can be reduced to a form with zero dynamics by some nondegenerate linear state-space transform ([4]):

$$\begin{cases} x^{t+1}=Ax^t+Bu^t \\ y^t=Cx^t, \qquad t=0,1,2,\dots, \end{cases}$$

where

$$A=\left(\begin{array}{cccccc} A_{11} & A_{12} & \cdots & A_{1l} & | & \tilde{A}_1 \\ A_{21} & A_{22} & \cdots & A_{2l} & | & \tilde{A}_2 \\ \cdots & \cdots & \cdots & \cdots & | & \cdots \\ A_{l1} & A_{l2} & \cdots & A_{ll} & | & \tilde{A}_l \\ --- & --- & --- & --- & --- & --- \\ \bar{A}_1 & \bar{A}_2 & \cdots & \bar{A}_l & | & \bar{\bar{A}} \end{array}\right),\quad B=\begin{pmatrix} B_1 \\ B_2 \\ \cdots \\ B_l \\ --- \\ 0 \end{pmatrix},$$

$$C=\begin{pmatrix} C_1 & C_2 & \cdots & C_l & | & 0 \end{pmatrix},$$

where

$$A_{ii}=\begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix}\in R^{r_i\times r_i};\ \bar{A}_j=\begin{pmatrix} 0 & \cdots & 0 & * \\ 0 & \cdots & 0 & * \\ \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & * \end{pmatrix}\in R^{(n-|r|)\times r_j};$$

$$A_{ij}=\begin{pmatrix} 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ 0 & 0 & \cdots & 0 & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}\in R^{r_i\times r_j},\ i\neq j;\ \tilde{A}_i\in R^{r_i\times(n-|r|)};\ i,j=1,\dots,l;$$

$$B_i=\begin{pmatrix} b_{i1} & b_{i2} & \dots & b_{il} \end{pmatrix},\ b_{ij}=\begin{pmatrix} \delta_{ij} \\ 0 \\ \cdots \\ 0 \end{pmatrix}\in R^{r_i\times 1},\ C_i=\begin{pmatrix} 0 & 0 & \cdots & 0 & H^i \end{pmatrix}\in R^{l\times r_i},$$

where H^i — i-th column of matrix H, δ_{ij} — denotes Kronecker delta.

Let us consider properties of column-wise RD. First, let us introduce a notion of incomplete (column-wise) relative degree (IRD).

Definition 3. A vector $r=(r_1,\dots,r_l)$ is the incomplete relative degree (IRD) vector if the first condition of Definition 1 is satisfied.

Let us formulate and prove some properties of relative degree (in what follows we consider column-wise relative degree). The first property is an invariance of RD definition's second condition under nonsingular output change that does not change system's IRD vector. Therefore, we can consider IRDs with the same set of components, but possibly different order of these components, equivalent, because any order can be achieved by permutation of columns of matrix B, i.e. permutation of outputs. Namely, whenever a nonslingular linear output transformation does not change IRD vector's set of components (i.e. it changes only the order of components), the second condition of Definition 1 is satisfied or not satisfied for both initial and transformed system simultaneously. The change of IRD vector's components order is acceptable, because it can be corrected by permutation of input matrix's columns, which does not change the absolute value of matrix's H (taken from the second condition of Definition 1) determinant.

Statement 1. Suppose input change $\widetilde{B} = B \cdot T$ ($T \in \mathbb{R}^{l \times l}, \det(T) \neq 0$) does not change the set of components of system's IRD vector (i.e. there exists a permutation of transformed system's IRD vector components such that after the permutation this vector coincides with the initial system's IRD). Then whenever $\det(CA^{r_1-1}B_1 | \dots | CA^{r_l-1}B_l) = 0$, we have $\det(CA^{r_1-1}\widetilde{B}_1 | \dots | CA^{r_l-1}\widetilde{B}_l) = 0$;if $\det(CA^{r_1-1}\widetilde{B}_1 | \dots | CA^{r_l-1}\widetilde{B}_l) = 0$, then $\det(CA^{r_1-1}\widetilde{B}_1 | \dots | CA^{r_l-1}\widetilde{B}_l) \neq 0$.

Proof. Note that by permutation of matrix's B columns we can sort IRD components descending without change of Definition 1 second condition. That's why without loss of generality we can assume that initial system and transformed system IRD vectors components are sorted descending:

$$r = (r_1^{(1)}, r_2^{(1)}, \dots, r_{p_1}^{(1)}, r_1^{(2)}, \dots, r_{p_2}^{(2)}, \dots, r_1^{(k)}, \dots, r_{p_k}^{(k)}),$$

where $r_i^{(s)} = r_j^{(s)} = r^{(s)}, i, j = 1, \dots, p_s, s = 1, \dots, k$ and $r^{(s)} > r^{(s+1)}, s = 1, \dots, k-1$.

Columns B_i are split into k groups, such that the values $r^{(s)}$ are the same for every component in the group.

1) Suppose $det(CA^{r_1-1}B_1 | \dots | CA^{r_l-1}B_l) \neq 0$, and an input transformation with matrix $T : \widetilde{B} = B \cdot T$ does not change IRD vector (sorted). Then columns $\widetilde{B}_i$ are also split into groups, similarly to groupsB_i. Let us show that $det(CA^{r_1-1}\widetilde{B}_1 | \dots | CA^{r_l-1}\widetilde{B}_l) \neq 0$. First, we show that whenever an input transformation does not change IRD vector, than matrix T have the following block upper triangular form:

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1k} \\ 0 & T_{22} & \cdots & T_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & T_{kk} \end{pmatrix}, \quad T_{ij} \in \mathbb{R}^{p_i \times p_j}.$$

As it was shown above, columns B are split into groups, and therefore matrix B has the form:

$$B = (B^{(1)}, B^{(2)}, \dots, B^{(k)}),$$

where

$$B^{(i)} = (B_{p_1 + \dots + p_{i-1} + 1}, \dots, B_{p_1 + \dots + p_i}).$$

Since after the transformation leading to $\widetilde{B}$ IRD vector does not change, we have

$$\widetilde{B} = (\widetilde{B}^{(1)}, \widetilde{B}^{(2)}, \dots, \widetilde{B}^{(k)}),$$

and the sizes of $\widetilde{B}^{(i)}$ and $B^{(i)}$ coincide. Then

$$\widetilde{B}^{(i)} = \sum_{j=1}^{k} B^{(j)} T_{ji}. \quad (5)$$

Let us show that $T_{qi} = 0$ for $q > i$. We fix an i and consider several cases where q is from k to $i+1$ (suppose that $i < k$, and therefore $r^{(i)} > r^{(k)}$).

Case $q = k$. We multiply equation (5) by $CA^{r^{(k)}-1}$. Since $r^{(k)} = min(r^{(j)})$, it follows that $CA^{r^{(k)}-1}B^{(j)} = 0$ whenever $j < k$. Since $i < k$, we have $CA^{r^{(k)}-1}\widetilde{B}^{(i)} = 0$, therefore $0 = CA^{r^{(k)}-1}\widetilde{B}^{(i)} = CA^{r^{(k)}-1}B^{(k)}T_{ki}$.

Since by assumption initial system has relative degree, matrix's $CA^{r^{(k)}-1}B^{(k)}$ columns are linearly independent. Therefore we get $T_{ki} = 0$.

Case $q = k-1$. We multiply equation (5) by $CA^{r^{(k-1)}-1}$. Taking into account $T_{ki} = 0$ and IRD definition, we get the following: $0 = CA^{r^{(k-1)}-1}\widetilde{B}^{(i)} = CA^{r^{(k-1)}-1}B^{(k-1)}T_{(k-1)i}$.

Since the initial system has relative degree, columns $CA^{r^{(k-1)}-1}B^{(k-1)}$ are linearly independent. Therefore we get $T_{(k-1)i}=0$. If we continue the procedure for $q=k-2,\dots,i+1$, we get $T_{qi}=0$ whenever $q>i$.

Let us show that

$$\widetilde{H}=(CA^{r_1-1}\widetilde{B}_1\,|\dots|\,CA^{r_l-1}\widetilde{B}_l)=(CA^{r_1-1}B_1\,|\dots|\,CA^{r_l-1}B_l)\cdot\overline{T}=H\cdot\overline{T},$$

where

$$\overline{T}=\begin{pmatrix}T_{11} & 0 & \cdots & 0\\ 0 & T_{22} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & T_{kk}\end{pmatrix}.$$

Lets introduce for matrixes $\widetilde{H}$ and H structure, analogous to $\widetilde{B}$ and B, i.e. $\widetilde{H}=(\widetilde{H}^{(1)},\dots,\widetilde{H}^{(k)})$, $H=(H^{(1)},\dots,H^{(k)})$. Then

$$\widetilde{H}^{(i)}=CA^{r^{(i)}-1}\widetilde{B}^{(i)}=CA^{r^{(i)}-1}(\sum_{q=1}^{k}B^{(q)}T_{qi})=CA^{r^{(i)}-1}B^{(i)}T_{ii},$$

since for $q>i$ $T_{qi}=0$, and for $q<i$

$$CA^{r^{(i)}-1}B^{(q)}=0.$$

Since $det(T)\neq 0$, so $det(T_{ii})\neq 0$ for all $i=1,\dots,k$. But this leads to

$$det(CA^{r_1-1}\widetilde{B}_1\,|\dots|\,CA^{r_l-1}\widetilde{B}_l)=det(CA^{r_1-1}B_1\,|\dots|\,CA^{r_l}$$

Since $det(\overline{T})\neq 0$ if and only if $det(T)\neq 0$, thus in case $det(CA^{r_1-1}B_1\,|\dots|\,CA^{r_l-1}B_l)\neq 0$ we obtain $det(CA^{r_1-1}\widetilde{B}_1\,|\dots|\,CA^{r_l-1}\widetilde{B}_l)\neq 0$. This proves first part of the assertion.

2) Suppose $det(CA^{r_1-1}B_1\,|\dots|\,CA^{r_l-1}B_l)=0$, and linear nondegenerate inputs transform with matrix $T:\widetilde{B}=B\cdot T$ has been performed, which doesn't change the IRD vector. Let us show that $det(CA^{r_1-1}\widetilde{B}_1\,|\dots|\,CA^{r_l-1}\widetilde{B}_l)=0$. Let's suppose that this is not right, and $det(CA^{r_1-1}\widetilde{B}_1\,|\dots|\,CA^{r_l-1}\widetilde{B}_l)\neq 0$. So, to receive contradiction we perform in the system $\{A,\widetilde{B},C\}$ inputs transform with matrix T^{-1}: $\overline{B}=\widetilde{B}\cdot T^{-1}$. This transform leads us to initial system, i.e. $\overline{B}=B$, so, IRD doesn't change during this transform. Then as it was proved above there must be $det(CA^{r_1-1}B_1\,|\dots|\,CA^{r_l-1}B_l)\neq 0$, but this is not right. Received contradiction proves the second part of assertion.

Thus it has been proved, that if the first condition of RD definition is fulfiled in different basises for the same IRD vector, then the second condition of RD definition is hold or not hold simultaneously in these basises. So certain vector defines a family of basises, where system has RD.

Lets prove one more statement about the properties of IRD and RD. Suppose, there exists basis of inputs manifold, where matrix B has the form B^*, and both conditions of RD are satisfied for vector r^*. Suppose that in some other basis α RD definition is not satisfied. Let's investigate possibility of linear transition from basis α to basis β by increasing some certain component of IRD vector. Without losing generality of argumentation, let's consider, that in all basises columns of inputs matrix are ordered nondecreasingly by the corresponding IRD components (this always can be achived by renumbering inputs and swapping columns of matrix). Result of this investigation plays significant role for constructing step-by-step algorithms for transition from arbitrary basis to a basis where RD definition is satisfied. This result is formulated in the following assertion.

Statement 2. Suppose that for given system $\{A,B,C\}$ RD definition is not satisfied. Suppose that for system $\{A,B^*,C\}$, where $B^*=B\overline{T},det(\overline{T})\neq 0$, RD definition is satisfied for vector $r^*=(r_1^*,r_2^*,\dots,r_l^*)$. Then if for system $\{A,B',C\}$, where $B'=B\widetilde{T},det(\widetilde{T})\neq 0$, the first condition of RD definition is satisfied for vector r', and $r_i'\leq r_i^*$ with at list one strict inequality among them, then some nondegenerate inputs transform for some index i^*, satisfying $(r')_{i^*}<r_{i^*}^*$, allows to increase some component of IRD vector r', which is less than $(r')_{i^*}$, to a value, greater or equal than $r_{i^*}^*$.

Proof. Without loosing generality of argument we shall suppose that columns in matrices B' and B^* are ordered undecreasingly by values of components of r' и r^*, correspondingly. Lets suppose that for system

$(')$ there doesn't exist transformation to some system $('')$, which satisfies the conditions of satement. Lets consider as a mentioned index i^* the maximum of possible values in the conditions of statement, i.e. such, that $r_i^* = (r')_i$ for $i > i^*$. Suppose there doesn't exist elementary transform of columns of matrice B', which increases some component of IRD with index not greater than i^*, up to the value $r_{i^*}^*$ or greater. By a condition of statement matrices $\overline{T}$ and $\widetilde{T}$ are square transform matrices and are nondegenerate. Lets consider the equality $B^* = B'(\widetilde{T})^{-1}\overline{T}$. For column i of this matrix equality it is fulfilled $B_i^* = B'T_i$, where T_i - i-th column of matrix $T = (\widetilde{T})^{-1}\overline{T}$. Multiplication of column T_i by matrix B' is equivalent to taking linear combination of it's columns with coefficients, equal to elements of column T_i, which can not be all zeros simultaneously because of nondegenerate matrix T. This linear combination may be represented by a sequence of elementary transformations of column $(B')_j$, if $T_{ij} \neq 0$. If matrix T is nondegenerate, than among it's elemnents $T_{ij}, i \leq i^*, j \geq i^*$ we can find at least one nonzero. Lets suppose that this element is $T_{i_0 j_0}$. There is correct equality $B_{j_0}^* = B'T_{j_0}$, which can be applied for nondegenerate transform of the column of matrix B' with index i_0 in order to obtain the coumn $B_{j_0}^*$. Thus in matrix B' by means of nondegenerate transform the column with index i_0 may be reduced to a form of the column j_0 of matrix B^*, keeping $i_0 \leq i^*, j_0 \geq i^*$, which provides the increasment of i_0-th component of IRD (as is demanded by the satement) thus making possible the transition from system $(')$ to system $('')$. That obtained contradiction finishes the proof of the statement. Moreover, the proof containes the way to know which certain component of IRD vector can be increased.

Notice 1. Lets note, that condition $(r')_i \leq r_i^*$ in the conditions of statement 2 doesn't limit the generality and is the natural correspondance between the ordered vectors od IRD and RD. It's easy to verify, since increassement of IRD component by means of linear transform is possible only when matrix H is degenerate.

Notice 2. Although it is not mentioned in the statement formulation, the proof is given without loosing generality for systems with ordered IRD vectors. Lets note that after transform procedure described in statement 2 initial system with ordered IRD vector may be reduced to a system with unordered IRD vector. In this case before repeatedly applying this statement the received system must be reordered by IRD components.

In particular, if system with IRD $(1,1,2)$ is reduced to a system with RD $(1,3,2)$ one need to understand that this reduction is equivalent to a transform to a system with RD $(1,2,3)$ from the point of ability to achieve such RD. Here is an example of such system:

$$\begin{cases} x_1^{k+1} = u_1^k + u_2^k \\ x_2^{k+1} = x_1^k + u_2^k \\ x_3^{k+1} = x_2^k \\ x_4^{k+1} = x_3^k \\ x_5^{k+1} = u_3^k \\ x_6^{k+1} = x_5^k \end{cases}; \begin{cases} y_1^k = x_1^k \\ y_2^k = x_4^k \\ y_3^k = x_6^k \end{cases}$$

$$\begin{matrix} CB_1 = [1\,0\,0]^T \Rightarrow r_1 = 1 \\ CB_2 = [1\,0\,0]^T \Rightarrow r_2 = 1 \\ CB_3 = [0\,0\,0]^T;\ CAB_3 = [0\,0\,1]^T \Rightarrow r_3 = 2 \end{matrix} \Rightarrow |H(1,1,2)| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.$$

IRD of this system is $r_{\gamma\delta\varepsilon} = (1,1,2)$. System doesn't have RD but can be reduced to it by a transform $\widetilde{u}_1^t = u_1^t + u_2^t, \widetilde{u}_2^t = u_2^t, \widetilde{u}_3^t = u_3^t$. As a result of such transform the second column of matrix B will be changed. For the transformed system we receive:

$$C\widetilde{B}_2 = [0\,0\,0]^T;\ CA\widetilde{B}_2 = [0\,0\,0]^T;\ CA^2\widetilde{B}_2 = [0\,1\,0]^T;\ | H(1,3,$$

This system has RD $r = (1,3,2)$. Renumbering of inputs allows to obtain ordered new RD vector: $r = (1,2,3)$.

The proved statement guarantees the existance of the sequence of column-wise elementary transforms, which reduces system with IRD to a form with RD if it exists.

In the development of step-by-step iterative algorithms it is an actual quesation to find the moment for stopping iterations. If we suppose that some algorithm is looking for basis of inputs in which system would satisfy the definition of relative degree using on each step increasement of some IRD component, it is necessary to have a condition, which guarantees that it is useless to continue the process of increasing IRD components. The answer to this question is given by the following statement concerning the maximum possible value of the RD vector compoment.

Statement 3. If for some system with order n RD definition is satisfied then all components of the RD vector are not greater than n.

Proof. Lets prove from the opposite. Suppose that some component of the RD vector is greater than n. In this case for some column of inputs matrix it is fulfilled $CB_j = 0, CAB_j = 0, CA^2B_j = 0, \dots, CA^{n-1}B_j = 0$. Applying Hamilton-Cayley theorem we can represent matrix A^n as a linear form of the previous degrees of matrix A, i.e. of the matrices $I, A, A^2, \dots, A^{n-1}$. In this case,

$$CA^n B_j = C(\alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \dots + \alpha_{n-1} A^{n-1}) B_j = \alpha_0 C$$

. Taking into consideration, that in similar way senior degrees (which are greater than n) of matrix A alse are represented using minor degrees, we obtain that equality $CA^k B_j = 0$ is valid for all k, and this leads to the invalidation of the first condition of the RD definition in this case. The received contradiction proves the statement.

3. Conclusion

On the basis of the proved properties of the RD we can suggest the following algorithm of the reduction of MIMO-system to a form with a relative degree by a nondegenerate linear transform of inputs. If for current basis RD definition is not satisfied, we will pass from one basis of matrix B to another as stated in Statement 2 trying to increase some IRD component. Lets consider all possible variants of such increasement in order to reach all vertexes of the graph, which correspond to all possible vectors of l components from $(1,1,\dots,1)$ to $(n,n,\dots,n)$. We start bypassing graph from the vertex with current IRD vector and move in the directions of increasement of IRD components. Edges of the graph connect its neighbouring vertexes (i.e. vertexes corresponding to vectors which differes in one component by a minimum possible whole number which allows transition between them via nondegenerate transform of inputs). According to Statement 1 when we reach vertex corresponding to the real RD vector, in the current basis both conditions of the RD definition are satisfied. In this case the success of the algorith is detected by verifying the second condition of RD definition in the current vertex. According to Statement 2 the path to such vertex (if the vertex exists) is available and can be found by search of all possible paths. According to Statement 3 we can stop search when the vertex with RD does not exist. This can be detected after complete search of all possible paths and verifying the second condition of the RD definition in all of the vertexes. The number of vertexes necessary to verify is limited by a value of $(n+l-1)!/(l!(n-1)!)$. Algorithm allows parallel computing on supercomputers and GPU systems.

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