On the Crossing Points of Circulant Graphs C(9, 3)

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Abstract: It is well known that determining the exact values of crossing number for circulant graphs is very difficult. Even so, some important results in this field are still proved. D.J. Ma was proved that the crossing number of $C(2m + 2, m)$ is $m + 1$[8]. Then such problem for $C(n, 3)$ was further solved [7]. Pak Tung Ho and X. Lin obtained accurate values for the crossover numbers of $C(3m, m)$ and $C(3m + 1, m)$[4][5]. In this paper, as a complement, we show that the edges from the principal cycle of $C(9, 3)$ do not cross each other in an optimal drawing.

Keywords: Crossing Number; Crossing Point; Circulant Graphs

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1. Introduction

All the graphs throughout this paper are simple. The terminology and notation are standard and can be found in [1, 10]. Here we repeat some definitions. A circulant graph $G = C(n; S)$ is the graph with the vertex set $V(C(n; S)) = \{v_i I 0 < i < n - 1\}$ and the edge set $E(C(n; S)) = \{v_i v_j I 0 < i < n - 1, 0 < j < n - 1, (i - j) \bmod n \in S\}$, $S \subset \{1, 2, \ldots, |n/2|\}$. When $S = \{1, k\}$, for some integer k, then $C(n; S)$ will be simplified to $C(n, k)$. For $C(n, 3)$, the cycle $C_n = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is called the principal cycle. It is clear that most types of circulant graphs are nonplanar.

A drawing of a graph is a representation in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. A drawing is good if it satisfies: (i) no edge crosses itself; (ii) no two edges cross more than once; (iii) no more than two edges cross at a common point; (iv) no edge crosses a vertex. A crossing is the common internal point of two edges. The crossing number of G, written $Cr(G)$, is the minimum number of crossings among all the good drawings of G in the plane. An optimal drawing of G is a good drawing whose number of crossings equals to $Cr(G)$. Obviously, G is planar if and only if $Cr(G) = 0$. And thus the crossing number is considered as an important topological measure for graphs. For a graph, computing the crossing number is NP-complete [2]. Therefore, the exact values of crossing numbers are known only for very restricted classes of circulant graphs. As far as we know, earlier results on this subject can be found [3]. Then H. Ren et al. [8] proved that the crossing number of $C(2m + 2, m)$ is $m + 1$. And such problem for $C(n, 3)$ was further solved[7]. Meanwhile, it was verified that the exact value of crossing number of $C(3m, m)$ is m, when $m > 3$ [5]. In 2007, this result was extended to $C(3m + 1, m)$ [4]. It is showed that $Cr(C(3m + 1, m)) = m + 1$ for $m > 3$. Even though these exact numbers are confirmed, the specific positions of these crossing points are still unknown. Moreover, little work can be found on this topic. In this paper, such position problem of crossing point in $C(9, 3)$ will be studied. In fact, the similar research for generalized Petersen graphs and Cartesian products is also presented in many papers. The interested reader may refer to [11, 12, 13] for more details.

The removal number of G, denoted by $h(G)$, is the smallest non-negative integer h such that removing some h edges from G results in a planar subgraph of G. Easy to see that removing $Cr(G)$ edges from an optimal drawing of G also yields a planar subgraph. Thus $Cr(G) > h(G)$. In [7], it was proved that $Cr(G) = h(G) = 3$, if $G = C(9, 3)$. It is a nice conclusion, because it implies that no edge is crossed twice in an optimal drawing of C(9, 3). However, where the three crossing points lie is not confirmed. That is, two distinct optimal drawings of C(9, 3) may have distinct crossing points. Whether such crossing points share some property is still unknown. In this paper, we show that no crossing in C(9, 3) is the result of a crossing between two edges which are from the principal cycle.

2. Basic lemmas and main result

In this part, we first introduce the preliminaries and the necessary lemmas for C(9, 3). Obviously, C(9, 3) maybe partitioned into the principal cycle C9 together with three vertex- disjoint subcycles c(0) = (v_0, v_3, v_6, v_0), c(1) = (v_1, v_4, v_7, v_1) and c(2) = (v_2, v_5, v_8, v_2). In fact, the edges (vi, vj), where i + 3 = j (mod 9) for i = 0, 1, . . . , 8, are also called chords of C9. For the convenience of the following discussion, we should color C(9, 3). Specifically, the edges in the principal cycle C9 are assigned blue color, and the chords are assigned red color. If a red edge crosses a red edge, we define this crossing as r - r crossing. The r - b crossing and b - b crossing are similarly defined.

Lemma 2.1 ([7]) For the circulant graph G = C(9, 3), h(G) = Cr(G) = 3.

As the discussion in introduction, a corollary of lemma 2.1 can be easily obtained as following.

Lemma 2.2 Let D be an optimal drawing of C(9, 3) . Then no edge is crossed twice.

Besides lemma 2.2, there is another obvious result needed to be mentioned.

Lemma 2.3 If p = (u,v, w) and q = (x,y, z) are any two cycles of length 3 in C(9, 3), then there is an automorphism θ such that θ(u), θ(v), θ(w) are x,y, z, respectively.

This result in lemma 2.3 is easy to verify, but plays an important role in the proof of the following main result.

Theorem 2.4 Let D be an optimal drawing of C(9, 3). Then D has no b - b crossing.

Obviously, theorem 2.4is a useful result because it gives the possible edges which destroy the planarity of C(9, 3). Furthermore, theorem 2.4's result also provides a facility for the research of topological structure of C(9, 3).

3. Proof of Theorem 2.4

Proof of Theorem 2.4. The proof is by reductio ad absurdum. Suppose that there exists at least one b - b crossing in D . Obviously, lemma 2.1 implies that D has at most three b - b crossings. Let $\{v_0, v_1, . . . , v_8\}$ be the vertex set of C(9, 3). The procedure will be partitioned into three cases. The following proof will use Jordan Curve Theorem.

Case 1. The optimal drawing D has three b - b crossings.

Then the principal cycle is divided into four parts (see figure 1). We may define such four parts as sections which are denoted by si(1 < i < 4) respectively. Furthermore, there exists a line (or a curve) l cutting every section into two halves (upper side and lower side).

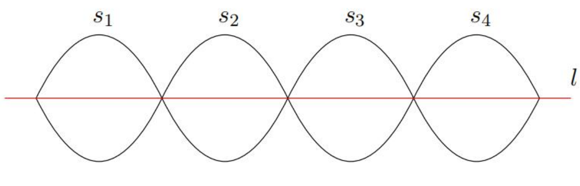

Fig. 1. Four sections of the principal cycle

By lemma 2.2, each side of si(i = 2, 3) contains at least one vertex (otherwise there is an edge which is crossed twice). And each sj(j = 1, 4) also contains at least one vertex, otherwise there is an edge which crosses itself and the drawing is not good. As a result, there are three vertices left. So we solve this problem through analyzing the positions of the remaining three vertices. For convenience, in the following discussion, we assume that v_0 lies on s_1. Note that there is no r - r crossing or r - b crossing in this case.

Since the three remaining vertices need to be distributed among four sections, at least one of the sections remains without additional vertices. If it is one of $\{s_1, s_4\}$, without loss of generality, say s_4, then figure 2(b) depicts the distribution of some vertices. Now v_0 needs to be joined to one of {vi, vi+1, vi+2}. If v_0 joins to vi, then $(v_0$, vi) crosses $(v_1$, vi+1). Moreover, ifv_0 joins to vi+1, then $(v_0$, vi+1) crosses $(v_1$, vi+2). Finally, ifv_0 joins to vi+2, then one of {$(v_0$, vi+2), $(v_1$, vi)} must cross $(v_2$,

v_{i+1}). All the possibilities produce an additional crossing, a contradiction.

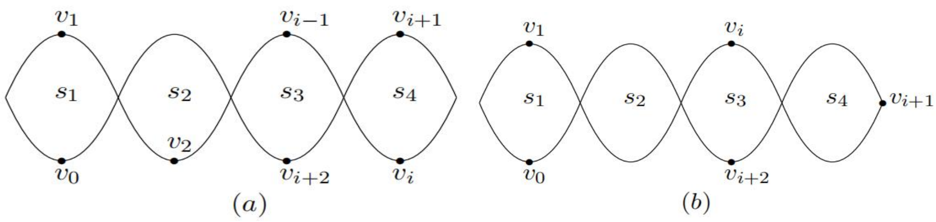

Fig. 2.

Thus, each of s_1 and s_4 must contain at least one of the remaining vertices. Then we can further claim that at least one remaining vertices does not lie on s_1 and s_4. Otherwise, one of s_1 and s_4 , say s_1 , contains three vertices. And then the distribution of vertices is shown in figure 3. Obviously, (v_0 , v_6) crosses (v_1 , v_7) which yields a contradiction.

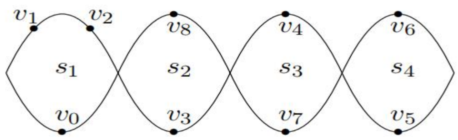

Fig. 3.

By the analysis above, the specific vertex-distribution is shown as figure 2(a). v_0 needs to be joined to one of $\{v_{i-1}, v_i, v_{i+1}\}$. If v_0 joins to v_{i-1} , then (v_0 , v_{i-1}) crosses (v_1 , v_i). Moreover, ifv_0 joins to v_{i+1}, then (v_0 , v_{i+1}) crosses (v_1 , v_{i+2}). Finally, ifv_0 joins to v_i, then one of $\{(v_0, v_i), (v_1, v_{i+1})\}$ must cross (v_2 , v_{i-1}). The additional crossings make it impossible that case 1 holds.

Case 2. The optimal drawing D has two b - b crossings.

In this case,the principal cycle is divided into three sections s_1 , s_2 , s_3 . Similar to the last case, there is at least one vertex on each side of s_2 , and each of $\{s_1, s_3\}$ has no less than one vertex. Here, we should notice that there may be a r - b crossing in case 2. It means that some chord crosses an edge of principal cycle. Without loss of generality, let $r_0 = (v_0, v_3)$ be this chord if such r -b crossing does exist. By lemma 2.3, we first determine the location of the vertices of c(0). Before the detailed analysis, we still assume that v_0 lies on s_1 , and then all the vertices of c(0) lying on s_3 cannot happen. In fact, it is also impossible that all the vertices of c(0) lie on s_1 . Otherwise, s_1 has at least seven vertices and s_2 has at most one. Therefore, we only discuss the three possible cases: the vertices of c(0) lie on s_1 , s_2 or s_1 , s_3 or s_1 , s_2 , s_3 .

Subcase 2.1 The vertices of c(0) lie on s_1 , s_2 .

Claim 1 v_3 can not lie on s_1 .

Proof of Claim 1. If v_3 also lies on s_1 , then v_6 lies on s_2 . And thus both two sides of s_2 are the possible sides where v_6 locates. We first consider that v_6 lies on the upper side of s_2. Then the vertex-distribution is as shown in figure 4(b). On the one hand, if there exists a r - b crossing in D, then, according to the front assumption, this r - b crossing lies on $r_0 = (v_0, v_3)$. As a result, the remaining chords should be drawn without any more crossing. Obviously, it is a contradiction, since (v_0 , v_6) crosses (v_2 , v_5). On the other hand, such r - b crossing does not exist, then (v_0 , v_6) crosses (v_2 , v_5) and (v_1 , v_4). Thus this drawing has at least four crossings. It is also a contradiction.

Furthermore, when v_6 lies on the lower side of s_2 , the vertices v_4 , v_5 and v_6 in figure 4(b) should be replaced by v_6 , v_7 and v_8 , respectively. Applying the same way above, we can also get contradictions.

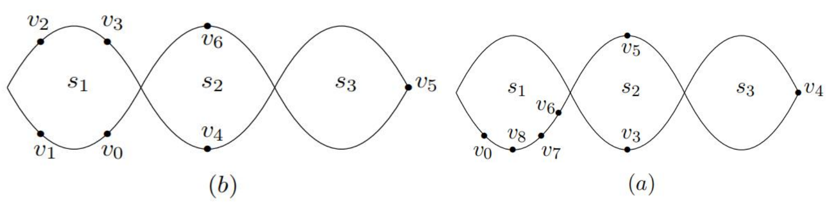

Fig. 4. The vertex-distribution for claim 1 and claim 2

Claim 1 implies that v_3 lies on s_2. However, the exact position of v_3 is still uncertain. Therefore, we assume that v_3 lies on the lower side of s_2 in the following discussion for subcase 2.1. Note that if v_3 lies on the upper side of s_2, then we may use the same way to solve this problem. Next we need to study the position of v_6.

Claim 2 v_6 can not lie on s_1.

Proof of Claim 2. If v_6 also lies on s_1, then the possible vertex-distribution is shown in figure 4(a). It is determined by the order in the subscripts of vertex. Similarly, if D has no r - b crossing, then (v_0, v_3) crosses (v_5, v_8) and (v_4, v_7). It is a contradiction that this drawing has at least four crossings. We now treat the case that $r_0 = (v_0, v_3)$ has a r - b crossing. The order in the subscripts of vertex implies that v_1 lies on the left part of v_3. Specifically, if v_1 lies on s_1, then (v_1, v_4) needs to cross (v_5, v_8). It is another crossing, a contradiction. And if v_1 lies on s_2, then (v_1, v_4) crosses (v_2, v_8). It is the fourth crossing which contradicts with the front hypothesis. □

Claim 2 further presents that v_6 lies on s_2 too. So the following process will be divided into two cases. First, v_3 and v_6 lie on the same side of s_2 (as shown in figure 5(a)). As a result, the position of c(0) is confirmed. It is straightforward that (v_0, v_6) crosses (v_4, v_7). This implies that D has no r - b crossing. Then we easily check that there is at least one crossing which is produced by two chords of $\{(v_3, v_6), (v_4, v_7), (v_5, v_8)\}$. This results in at least four crossings in D which is a contradiction.

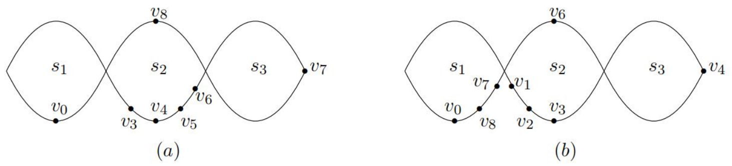

Fig. 5.

Therefore, we only need to consider the final case that v_6 lies on the upper side of s_2. The specific vertex-distribution is shown in figure 5(b). Easy to find that s_3 contains at least one of $\{v_4, v_5\}$. Thus we may assume that v_4 lies on s_3. Moreover, by the order in the subscripts of vertex, v_1 's position is on the left part of v_3. v_7 also lies on the left part of v_6. Here, it is necessary to say that the exact positions of v_1 and v_7 are still uncertain. Similar to the proof above, we first suppose that (v_0, v_3) has no r -b crossing. The following discussion will proceed through analyzing v_1 's exact position.

(i) If $v1$ lies on $s1$, then $(v0, v6)$ crosses $(v1, v4)$;

(ii) If $v1$ lies on $s2$ and $v7$ lies on $s2$, then $(v0, v6)$ crosses $(v4, v7)$;

(iii) If $v1$ lies on $s2$ and $v7$ lies on $s1$ (as shown in figure 5(b)), then $(v2, v8)$ crosses $(v1, v4)$.

All the three crossings have been found in each case above. Easy to see it is impossible to add the remaining chords without getting additional crossings. It is a contradiction. Not only that, the final subcase that D has no r - b crossing can be also solved through the similar discussion according to (i) - (iii). The proof for subcase 2.1 ends.

Subcase 2.2 The vertices of c(0) lie on s_1, s_3.

The discussion for this subcase is almost the same to that in subcase 2.1. Therefore, we will briefly highlight the important point of this proof.

Claim 3 It does not happen that both v_3 and v_6 lie on s_3.

Proof of Claim 3. If such situation happens, then the positions of $\{v_3, v_4, v_5, v_6\}$ can be determined as shown in figure 6(a). Since vj needs to connect one of $\{v_4, v_5\}$ to form the chord rj, there exists no r - b crossing on $r_0 = (v_0, v_3)$, otherwise (v_0, v_6) and rj can not be drawn without an additional crossing. Furthermore, vi also needs to connect one of $\{v_4, v_5\}$ to form a chord, denoted by ei. Then (v_0, v_3) crosses ei. The analysis above shows this drawing has at least four crossings. It is a contradiction.

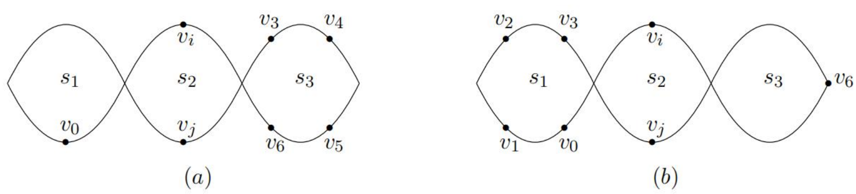

Fig. 6. The vertex-distribution for Subcase 2.2.

By Claim 3, v_3 and v_6 lie on s_1, s_3, respectively. Without loss of generality, suppose that v_3 lies on s_1 and v_6 lies on s_3. Figure 6(b) presents the specific vertex-distribution. Then vi needs to connect one of $\{v_1, v_2\}$ to form the chord ri. Similarly, vj also needs to connect one of $\{v_1, v_2\}$ to form a chord, denoted by ej. Now we easily check that drawing ri, ej, (v_0, v_6) and (v_3, v_6) yields at least two crossings whether $r_0 = (v_0, v_3)$ has a r - b crossing or not. It is a contradiction, and thus the proof for subcase 2.2 is over.

Subcase 2.3 The vertices of c(0) lie on s_1, s_2, s_3.

For convenience, let v_3 be on s_2 and v_6 be on s_3. Obviously, it is also possible that v_3 lies on s_3 and v_6 lies on s_2, but the proof techniques for such two cases are the same. Thus the following discussion focuses on the former case. By the order in the subscripts of vertex, the upper side of s_2 contains at least one of $\{v_7, v_8\}$, and then further let v_7 be that one.

Claim 4 It does not happen that both v_1 and v_4 lie on s_2.

Proof of Claim 4. If such situation happens, then the exact positions of vertices can be determined except for $\{v_5, v_8\}$ (see figure 7(a)). v_5 is on $v_4 \to v_6$ segment and v_8 is on $v_7 \to v_0$ segment. We first treat the case that the drawing D has a r - b crossing on $r_0 = (v_0, v_3)$. Then (v_0, v_6) and (v_3, v_6) can be drawn as in figure 7(a). As a result, (v_1, v_7) and (v_4, v_7) have to lie in the interior of s_2, otherwise such two chords cross (v_0, v_6) which yields additional crossings. Observe the structure and easily find that (v_2, v_5) must cross one of $\{(v_0, v_6), (v_3, v_6), (v_4, v_7)\}$. It denies the possibility that $r_0 = (v_0, v_3)$ has the r - b crossing. Furthermore, checking all the possible positions of v_8 shows that one of $\{(v_2, v_8), (v_5, v_8)\}$ also contains a crossing. This crossing, together with that on (v_2, v_5), may result in a contradiction. □

Claim 4 tellsusan important fact that at least one of $\{v_1, v_4\}$ does not lie on s_2. Without loss of generality, suppose that v_1 lies on s_1. The specific vertex distribution is shown as figure 7(b). Similarly, the crossing between (v_1, v_4) and (v_0, v_6) ensures that there is no r - b crossing on $r_0 = (v_0, v_3)$ in D. Then $r_0 = (v_0, v_3)$ can be drawn as that in figure 7(b). Moreover, v_4 has to join to v_7 in the interior of s_2 to avoid crossing (v_0, v_6). Now we check all the possible drawing of (v_2, v_5) and find that there must be a crossing on (v_2, v_5). Therefore, we get another two crossings besides two b - b crossings, a contradiction. The proof for subcase 2.3 ends.

Up to now, all the proof of case 2 have been finished.

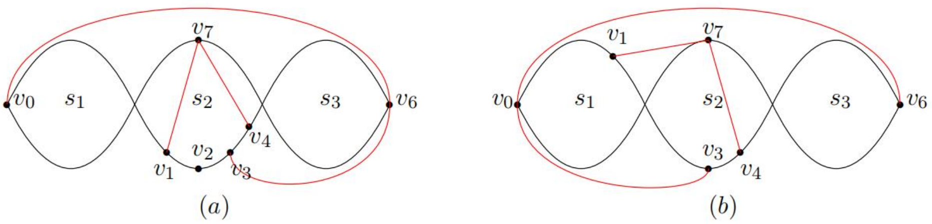

Fig. 7. The vertex-distribution for subcase 2.3.

Case 3. The optimal drawing D has one b - b crossing.

The principal cycle is divided into two sections, denoted by s_1, s_2. Obviously, there must be a section of $\{s_1, s_2\}$ which has at least five vertices. Thus it is reasonable to assume that s_1 contains at least five vertices. We study case 3 from the aspect of the number of vertices which lie on s_1. Here, we still need to notice that there may be r - b crossings in this case. As the discussion in case 2, suppose that v_0 lies on s_1. In the following discussion, we will give the detailed proof for the subcase that s_1 contains five vertices. The other subcases can be also treated by almost the same way.

A possible vertex-distribution that s_1 contains five vertices is shown in figure 8. If D has no r - b crossing, then (v_3, v_6) may be drawn as figure 8. It is not hard to see that (v_3, v_6) crosses (v_1, v_7) and (v_2, v_8). However, lemma 2.2 tells that one edge can not be crossed twice in D. To avoid this contradiction, there must be at least one edge of $\{(v_3, v_6), (v_1, v_7), (v_2, v_8)\}$ which needs to change the drawing. It means that at least one of $\{(v_3, v_6), (v_1, v_7), (v_2, v_8)\}$ has a r - b crossing.

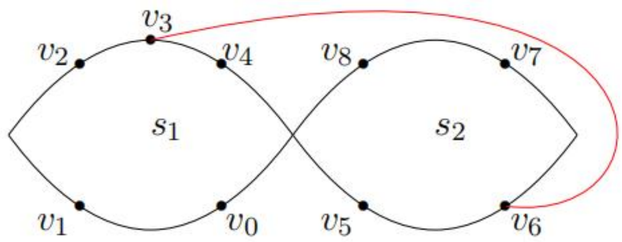

Fig. 8. The vertex-distribution for case 3.

Claim 5 (v_3, v_6) has a r - b crossing.

Proof of Claim 5. This proof is also by reductio ad absurdum. If (v_3, v_6) has no r - b crossing, then at least one of $\{(v_1, v_7), (v_2, v_8)\}$ crosses an edge of principal cycle to avoid crossing with (v_3, v_6) (see figure 8). Obviously, it is impossible that both (v_1, v_7) and (v_2, v_8) have r - b crossings. Otherwise, the drawing of (v_0, v_6) and (v_2, v_5) may yield the fourth crossing. Thus we further assume that (v_1, v_7) has a r - b crossing but (v_2, v_8) does not. Then (v_3, v_6) still crosses (v_2, v_8), and (v_0, v_6) crosses (v_2, v_5). It is a contradiction. □

Claim 5 has shown that D has ar - b crossing on (v_3, v_6). In fact, we can further claim that D has another r - b crossing. Otherwise, (v_2, v_5) crosses (v_0, v_6) and (v_1, v_7) which contradicts with lemma 2.2. So (v_2, v_5) have to crosses an edge of principal cycle to avoid crossing with (v_0, v_6) and (v_1, v_7). It is essential to explain why this r - b crossing can not lie on (v_0, v_6) or (v_1, v_7). Because if one chord of $\{(v_1, v_7), (v_0, v_6)\}$ has a r -b crossing, then the crossing between (v_2, v_5) and the other chord of $\{(v_1, v_7), (v_0, v_6)\}$ becomes the fourth one, a contradiction. So far, the three crossings of D have been found. However, (v_2, v_8) still crosses (v_4, v_7) which yields an additional crossing. Thus D can not have one b - b crossing when s_1 contains five vertices. The remaining subcases that s_1 contains 6, 7 or 8 vertices may be also verified through the same way. The proof for case 3 is over.

In summary, all the discussion above verifies the result of main theorem.

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Pak Tung Ho and X. Lin the exact values of the crossing numbers of C (3m, m) and C (3m + 1, m) are obtained.