Nonexpansive Mappings in Fixed-Point Iterations and Their Applications in Game Theory

Yufan Zou

Chongqing Normal University, Chongqing 401331, China

Abstract: This paper explores the theoretical foundation of non-expansive mappings in fixed-point iteration and its specific applications in game theory. As a mapping that maintains or reduces the distance between elements, non-expansive mappings provide an effective mathematical tool for finding stable strategies in dynamic games through their inherent convergence properties. The article first introduces the basic concepts of fixed-point theory and defines non-expansive mappings, then analyzes their application in solving Nash equilibria in game theory, with a particular emphasis on the stability and efficiency of iterative methods in the solution process. Additionally, through mathematical modeling and case analysis, this study demonstrates the practical effects and potential applications of non-expansive mappings in multi-stage games and complex strategy updates. This research not only enhances the understanding of fixed-point iteration methods in both theory and practice but also offers new perspectives and methods for addressing high-dimensional strategy spaces in game theory.

Keywords: Fixed-Point Iteration; Non-Expansive Mapping; Game Theory

1. Introduction

The fixed-point iteration method is a numerical technique used to solve fixed points of functions, with wide applications in modern mathematical analysis, economics, physics, and other fields. By repeatedly applying a mapping, the fixed-point iteration method gradually brings an initial point closer to a stable point, known as the fixed point. Non-expansive mappings, as a special type of mapping, possess the property of maintaining or reducing the distance between points, which endows them with excellent convergence properties in fixed-point iterations. Compared to expansive mappings, non-expansive mappings effectively prevent instability in the iterative process, ensuring that the system converges to a fixed point within a finite number of steps.

Many problems in game theory can be transformed into the problem of finding fixed points, particularly in solving dynamic games and Nash equilibria[1]. The fixed-point theorem provides the theoretical foundation for proving the existence of Nash equilibria, while non-expansive mappings offer an effective iterative tool for progressively approximating equilibrium solutions in game theory. By introducing the fixed-point iteration method of non-expansive mappings, the solutions to complex games can be simplified, and the stability and convergence of the solutions can be guaranteed. Therefore, this paper aims to systematically study the fixed-point iteration method of non-expansive mappings and its applications in game theory, revealing the potential and advantages of this method in solving complex problems in game theory.

2. Overview of Fixed-Point Theory

Fixed-point theory is a core concept in mathematics, widely applied to solving nonlinear equations, optimization problems, and equilibrium analysis in game theory[2]. A fixed point refers to a point that remains unchanged under a mapping, meaning that the result of the mapping is the point itself. By studying these special points, we can simplify the solution process for many complex problems, particularly playing a crucial role in iterative algorithms.

In fixed-point theory, the contraction mapping theorem is a key result, which states that certain types of mappings must converge to a unique fixed point. This result provides the theoretical foundation for solving fixed points through iteration and serves as the core principle behind many numerical methods. Within this framework, by repeatedly applying a mapping, the iterative process gradually stabilizes, approaching the fixed point. This method can be applied to solving various mathematical problems, particularly those involving stable states.

In game theory, fixed-point theory holds special significance, especially for proving and solving Nash equilibria. A Nash equilibrium can be viewed as a fixed point in the strategy space of game participants, and studying these fixed points helps to understand the stable strategies in complex games. In multi-stage or dynamic games, the continuous adjustment and updating of strategies can be realized through the fixed-point iteration method, making the theoretical analysis more concise and intuitive. Therefore, fixed-point theory not only provides a powerful tool for mathematical analysis but also offers new perspectives and solutions to many core issues in game theory.

3. Definition and Characteristics of Non-Expansive Mappings

A non-expansive mapping is a term used in mathematics to describe a specific type of mapping that does not increase the distance between input elements during processing[3]. More technically, if a mapping T acts on any two points x and y in a metric space, and the distance between the mapped points T(x)andT(y) is no greater than the original distance between x and y, then this mapping is called non-expansive. This property is crucial because it ensures that, during iteration, the sequence of solutions does not become unstable due to expansion, making the solution process more controlled and predictable.

In game theory, the application of non-expansive mappings is especially important. Game theory often involves finding a stable point for participants' strategies, known as the equilibrium state. In such scenarios, non-expansive mappings offer a powerful tool, as they maintain or reduce the "distance" between participants' choices during the iteration process, gradually guiding the system toward equilibrium. For example, in the case of repeated games or dynamic strategy adjustments, using non-expansive mappings can effectively prevent instability in the strategy adjustment process, thereby aiding in the discovery of stable strategies or Nash equilibria in games.

One key advantage of non-expansive mappings is their universality and flexibility. Compared to other more restrictive types of mappings (such as contraction mappings), non-expansive mappings do not require that the distance be reduced during the mapping process; they only need to ensure that the distance does not increase. This relaxed condition allows non-expansive mappings to be applied in a broader range of situations, particularly in complex systems where ensuring contraction conditions is difficult. Moreover, the combination of theory and practice for this type of mapping provides a strong theoretical foundation for understanding and designing stable iterative algorithms.

Thus, non-expansive mappings have demonstrated their unique value in both theoretical research and practical applications. When exploring complex economic models, ecological system dynamics, and various equilibrium problems in social sciences, non-expansive mappings are an indispensable tool. Especially in game theory, their ability to stably solve equilibrium points through iterative methods makes them an important mathematical tool for addressing such problems.

4. Applications of Non-Expansive Mappings in Game Theory

4.1 Nash Equilibrium and Non-Expansive Mappings

Nash equilibrium is a core concept in game theory, describing a strategy combination in which no participant can unilaterally change their strategy to improve their own benefit. The application of non-expansive mappings here lies in their stable iterative properties for finding such equilibrium points. In some game models, the strategy update rules can be constructed as non-expansive mappings, ensuring that, starting from any initial strategy, the iterative process of strategy updates converges to the Nash equilibrium.

4.2 Applications in Dynamic Games

In dynamic games, participants' strategies constantly change, requiring continuous updates to respond to the actions of other participants. Due to their inherent convergence properties, non-expansive mappings are well-suited for modeling such problems, ensuring that each step of strategy adjustment does not deviate from the final stable state, thereby effectively approaching the dynamic equilibrium solution.

4.3 Simplifying Complex Game Models

In multi-stage games or complex games with high-dimensional strategy spaces, non-expansive mappings can simplify the computational process. By transforming complex strategy update rules into non-expansive mappings, numerical methods can be efficiently employed to compute equilibrium strategies, which is valuable for both theoretical research and practical applications.

4.4 Stability Analysis in Game Theory Models

The inherent properties of non-expansive mappings can be used to analyze the stability of game models. For example, by analyzing the non-expansive nature of strategy update rules in a game model, it is possible to predict the long-term behavior of the model, such as whether it will converge to an equilibrium point and the stability of that equilibrium.

4.5 Algorithm Design and Optimization

In computational applications of game theory, designing effective algorithms is key to solving practical problems. Non-expansive mappings not only provide a theoretical foundation for algorithm design but also offer direction for algorithm optimization due to their excellent convergence properties. By rationally designing iterative algorithms based on non-expansive mappings, it is possible to handle a wider range of strategy updates and optimization problems while ensuring both convergence speed and computational accuracy.

5. Convergence Analysis of Fixed-Point Iteration for Non-Expansive Mappings

In many fields of mathematics and applied mathematics, the fixed-point iteration method is an important tool for solving nonlinear equations and system stability problems. Particularly in game theory, the fixed-point iteration method for non-expansive mappings is widely used to find stable solutions for various game models, such as Nash equilibria. The core characteristic of non-expansive mappings is that the output of the mapping does not spread further than the input. This property ensures that iterative sequences, starting from any initial point, converge to one or more fixed points.

Theoretically, the convergence of non-expansive mappings is based on their distance non-expansion property. This means that each iteration does not increase the distance between iterated values, thus avoiding solution divergence and ensuring the stability of the iterative process. Moreover, under certain non-expansive conditions and appropriate space properties, such as completeness, the convergence of the sequence to a fixed point can be guaranteed, even in high-dimensional and complex strategy spaces.

In applications to game theory, the use of the fixed-point iteration method for non-expansive mappings not only efficiently solves Nash equilibria but also allows for the analysis of equilibrium stability and robustness. For instance, in models from economics or social sciences, the path of iteration to the equilibrium point reflects the adjustment and adaptation mechanisms in actual dynamic processes, providing deep insights into the strategy adjustment process. Therefore, an in-depth study of the convergence of non-expansive mapping iterations is not only a theoretical necessity but also offers valuable guidance for practical applications.

6. Conclusion

This paper explores the application of non-expansive mappings in the fixed-point iteration method, and analyzes its specific role in solving equilibrium in game theory, particularly in dynamic games. By systematically elaborating on the properties of non-expansive mappings and conducting a convergence analysis, we have demonstrated its great potential in stably solving strategy equilibria. Future research could further explore its applications in complex systems and develop new computational methods to address the challenges posed by complex games.

References

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About the author:

Yufan Zou，1999-，Female, Han,Chongqing, Master of Science, Variational analysis