Financial Statistical Journal 2024, 7(1), 5632.
https://doi.org/10.24294/fsj.v7i1.5632

Article

Estimating parameters of the CAPM under generalised asymmetric student-*t* distribution—The case of the Warsaw Stock Exchange sectoral indices

Mateusz Pipień

Cracow University of Economics, 31-510 Kraków, Poland; Pipienm@Uek.Krakow.Pl

CITATION

Pipień M. Estimating parameters of the CAPM under generalised asymmetric student-*t* distribution—The case of the Warsaw Stock Exchange sectoral indices. Financial Statistical Journal. 2024; 7(1): 5632. https://doi.org/10.24294/fsj.v7i1.5632

ARTICLE INFO

Received: 3 April 2024
Accepted: 21 May 2024
Available online: 30 May 2024

COPYRIGHT

Copyright © 2024 by author(s). *Financial Statistical Journal* is published by EnPress Publisher, LLC. This work is licensed under the Creative Commons Attribution (CC BY) license. https://creativecommons.org/licenses/by/4.0/

Abstract: This paper analyses selected sub-indices listed on the Warsaw Stock Exchange (WSE) covering seven sectors: construction, IT, media, real estate, fuel, food, and telecommunications, from 3 January 2006 to 29 May 2020. We use daily, weekly, monthly, and quarterly data, resulting in 3600 daily, 751 weekly, 172 monthly, and 56 quarterly observations. The WIG index quotations were used to approximate the market portfolio and the Poland 10Y government bond yields for the risk-free rate. We have estimated the parameter β in CAPM regression using three different stochastic assumptions for the error term. The basic stochastic framework of the model utilises the generalised asymmetric student-*t* distribution (GAST). We have also estimated the parameter β based on the symmetric version of the GAST distribution and on the Gaussian one. These models can be treated as special cases of the basic framework. The estimates of the β parameter are influenced by the assumptions made about the error term. The data indicates that except for WIG-Paliwa, the Gaussian error term leads to larger β estimates than other non-Gaussian specifications. The inference about the shape parameters is not very certain, and the data does not strongly support the two-piece mechanism that enforces the asymmetry of the error term distribution. Furthermore, the estimates of the β parameter depend strongly on the frequency of the analysed series.

Keywords: CAPM; asymmetric *t*-distribution; Warsaw Stock Exchange
JEL Classification: C52; C53; C58

1. Introduction

142 years ago, the introduction of probability distributions that differed from the Gaussian case occurred. This can be seen in the works of De Forest [1,2], Edgeworth [3], and Pearson [4,5]. In Genton's book [6], several empirical examples were analysed to illustrate that there has been a quest for more flexible families of distributions than the Gaussian to serve as a sampling model, but this has not been achieved. Over the last four decades, there has been a desire to go beyond normality, as stated in the subtitle of Genton's [6] book, particularly to properly model the empirical distribution of data that represents fluctuations observed in financial markets.

Over the last fifty years, theoretical investigations in finance often assumed that the distributions of financial returns follow a Gaussian distribution. For example, the theory of pricing a European option explicitly assumes normality, as seen in the works of Black and Scholes [7] and Merton [8]. However, the theory of mean-variance portfolio selection, developed by Markovitz [9], does not require Gaussian returns. Still, it does assume that the distribution of asset returns can be characterised by their first and second moments. As a result, the utility function considered by a decision maker is a function dependent only on the mean and variance of portfolio returns.

The Capital Asset Pricing Model (CAPM), developed by Sharpe [10] and

Treynor [11] and further advanced by Lintner [12] and others, stands out as one of the most significant outcomes of the Markovitz theory. Applications of the CAPM yield valuable insights into the risk associated with financial assets. The widespread adoption of the CAPM is primarily attributed to its ability to estimate the cost of equity (COE) capital for a firm or industry within a robust theoretical framework, as highlighted by Graham and Harvey [13]. The COE, representing the return percentage offered by a company or industry to its equity shareholders based on their investments, serves as a critical gauge for investors evaluating investment profitability. Subpar returns may prompt investors to pursue opportunities with higher potential returns. Furthermore, accurate COE estimations are essential for various reasons in industries that significantly impact the economy. In the banking sector, Altavilla et al. [14] emphasised that COE provides essential information for financial stability and the assessment of monetary policy transmission. Consequently, COE serves as a noteworthy measure not only for investors but also for regulatory and supervisory institutions.

Kaplan and Peterson [15] explain that the CAPM-driven approach is popular among practitioners, especially investors, because there are no theoretically justified alternatives. However, in the empirical literature two decades ago, a consensus was established that the CAPM does not hold. As a result, CAPM-β cannot explain expected returns. Fama and French [16] reviewed various attempts to solve this issue. Andrei et al. [17] recently investigated why investors are still reluctant to give up the model despite the widespread rejection of the CAPM theory from an empirical point of view. They propose a theoretical framework that suggests the Capital Asset Pricing Model (CAPM) holds from the investors' perspective. However, it fails to match the empirical evidence in one specific aspect: the securities market line (SML) appears empirically flat but occasionally becomes steeper, for example, when public information reduces investor disagreement. The study explains the results by Savor and Wilson [18], who found a strong correlation between expected returns and CAPM-β on days when news about inflation, unemployment, or FOMC interest rate decisions is scheduled to be announced.

The assumption that financial returns are normally distributed is often assumed in the existing literature on empirical asset pricing, despite evidence showing that this assumption is not empirically relevant. Since the 1960s, empirical analyses have largely failed to support this normality assumption. This has significant implications for the Capital Asset Pricing Model (CAPM). Researchers such as Black [19], Fama and French [20], Zhou and Yin [21], Wei [22], and Urom et al. [23] have found that the results of estimating the CAPM-β parameters can vary across different models, affecting the estimated beta values and the relationship between risks and market returns.

Adcock [24] states that it's unnecessary to assume that returns follow a normal distribution to satisfy the CAPM axioms. He found that the family of skew-normal distributions (described in Azzalini and Dalla Valle [25]) is empirically important for UK stocks, and estimates of the CAPM β parameter change significantly when the sampling distribution allows for skewness. Adcock and Shutes [26] previously analysed the impact of assuming a student-t distribution for returns on the CAPM β estimates and found substantial effects on the estimation outcomes. These two papers

are part of a wide range of research on the empirical relevance of the CAPM theory, considering non-standard stochastic assumptions that explicitly allow for heavy tails and/or asymmetry in the underlying distributions. Other works by Affleck-Graves [27], MacKinlay and Richardson [28], Zhou [29], Harvey and Siddique [30], Li [31], Theodosiou and Theodossiou [32], and more recently, Bao et al. [33], are examples of extensive research on the application of flexible non-Gaussian families of distributions in the problem of risk assessment using the CAPM.

In this paper, we analyse the Capital Asset Pricing Model (CAPM) using a variety of distributions that go beyond the normal family. Specifically, we employ distributions from the generalised asymmetric student-t class developed by Harvey and Lange [34], which accounts for skewness and asymmetric tail thickness. Furthermore, the generalisation by Harvey and Lange [34] combines the student-t family of distributions with the Generalized Error Distributions (GED) class in a streamlined parametrisation. Our study focuses on the Warsaw Stock Exchange (WSE) indices. It examines the estimates of the β parameter for a selection of sector indices, including fuel (WIG-Paliwa), food (WIG-Spozywczy), telecommunications (WIG-Telekom), computers (WIG-Info), real estate (WIG-Nieruchomosci), media (WIG-Media), and construction companies (WIG-Budowlany) listed on the WSE. In our econometric analyses, we follow the main goals of the paper stated as:

1) Analysis of the sensitivity of the CAPM β parameter estimates under a family of distributions allowing for heavy tails, skewness, and tail asymmetry.
2) Analysis of the sensitivity of the performance of the CAPM to the frequency of analysed data (daily, weekly, monthly, and quarterly).

In the following paper, we will first provide an overview of the generalised asymmetric student-t distribution proposed by Harvey and Lange [34]. We will discuss the symmetric version of the H&L distribution and the skewed and tail-asymmetric cases, each as separate sections within the generalised family. Chapter 3 will review some econometric strategies concerning the assumptions of the error term in the CAPM regression model. Finally, Chapter 4 will describe the data, the model framework, and the main empirical results.

2. Parametric representation of skewness and tail asymmetry

In a paper by Li and Nadarajah [35], it is mentioned that student's t distribution and its generalisations have become the most popular models for economic and financial data. Our work will focus on analysing some of the generalisations of the student-*t* distribution proposed by Harvey and Lange [34]. These generalisations unify the family with the Generalized Error Distributions (GED), and the final construct goes beyond previous results by Zhu and Zinde-Walsh [36], Zhu and Galbraith [37], Fernández and Steel [38], and Theodossiou [39], among others. For more information, we refer to the literature overview and discussion by Harvey and Lange [34].

Previous empirical analyses have demonstrated the extraordinary flexibility of generalised families. For instance, Mazur and Pipień [40] studied TV-GARCH models with conditional distribution proposed by Zhu and Galbraith [37], while Mazur and Pipień [41] investigated coordinate-free multivariate distributions obtained based on Harvey and Lange [34] asymmetric-t class.

Consider a random variable with zero mode and unit scale z with the following density, parametrised by $\eta > 0$ and $\upsilon > 0$:

$$p(z|\eta,\upsilon) = K(\eta,\upsilon)\left(1+\frac{1}{\eta}|z|^{\upsilon}\right)^{-\frac{\eta+1}{\upsilon}} \tag{1}$$

where the normalising constant $K(\eta, \upsilon)$ is given by the formula:

$$K(\eta,\upsilon) = \frac{\upsilon}{2\eta^{\frac{1}{\upsilon}}B(\frac{1}{\upsilon},\frac{\eta}{\upsilon})}$$

and $B(.,.)$ is the beta function; see Harvey and Lange [34]. The parameter $\upsilon > 0$ determines the shape around the mode, with $\upsilon = 2$ leading to the student-*t* type shape. The parameter $\eta > 0$ controls the tail behaviour and determines the existence of moments only in the case where $\upsilon = 2$. The advantage of the unification proposed by Harvey and Lange [34] is that it includes a list of well-known distribution families and offers elasticity by parametrising a very broad class of distributions. This links the properties of the student-*t* and GED families in an unconstrained but continuous form. In particular, when $\eta \to +\infty$, the broad GED with shape parameter υ is defined. For $\upsilon = 1$, the Laplace distribution can be obtained, and $\upsilon = 2$ defines the Gaussian distribution, provided that $\eta \to +\infty$.

To introduce skewness in a distribution, one may use inverse scale factors as explained in Ferreira and Steel's [42] work. The resulting density accounts for possible skewness around the mode in a manner proposed by Fernández and Steel [38], and similarly by Hansen [43] for a student-*t* distribution:

$$p(z|\eta,\upsilon,\alpha) = K(\eta,\upsilon)\begin{cases}\left(1+\frac{1}{\eta}\left|\frac{z}{2\alpha}\right|^{\upsilon}\right)^{-\frac{\eta+1}{\upsilon}}, z \le 0\\ \left(1+\frac{1}{\eta}\left|\frac{z}{2(1-\alpha)}\right|^{\upsilon}\right)^{-\frac{\eta+1}{\upsilon}}, z > 0\end{cases} \tag{2}$$

Harvey and Lange [34] alternatively parametrised Equation (2) by $\alpha \in (0, 1)$, with $\alpha = 0.5$ restoring symmetry. The Arnold and Groeneveld [44] measure of skewness to the mode is referenced here. The distribution given by Equation (2) has an elegant property:

$$P(z \le 0) = \alpha \tag{3}$$

It is important to note that the distribution of *z*, defined by Equation (2), may no longer have a zero mean, even though it remains unimodal; see Harvey and Lange [34].

It is possible to disrupt the symmetry in Equation (1) by implementing a two-piece mechanism as demonstrated by Zhu and Galbraith [37] when they extended the student-*t* distribution. The resulting density distribution is a member of the two-piece skew family.

$$p(z|\eta_L,\upsilon_L,\eta_R,\upsilon_R) = K_{LR}\begin{cases}\left(1+\frac{1}{\eta_L}|z|^{\upsilon_L}\right)^{-\frac{\eta_L+1}{\upsilon_L}}, z \le 0\\ \left(1+\frac{1}{\eta_R}|z|^{\upsilon_R}\right)^{-\frac{\eta_R+1}{\upsilon_R}}, z > 0\end{cases} \tag{4}$$

where the normalising constant KLR is given by the formula:

$$K_{LR} = \frac{1}{\frac{0.5}{K(\eta_L, \upsilon_L)} + \frac{0.5}{K(\eta_R, \upsilon_R)}}$$

It is possible to consider a variable distributed according to the density in Equation (4) as a mixture of two distributions in Equation (1) with different parameters υ and η. These distributions are also renormalised to ensure that the resulting probability density function is continuous at the mode (which is zero). As with the construct defined by Equation (2), the expected value of the distribution given by Equation (4) may not be equal to zero (if it exists). The probability mass around the mode (which is zero) is not equally spaced. In their work, Harvey and Lange [34] provide the following function, which describes the probability that z is not positive:

$$P(z \leq 0) = \frac{K(\eta_L, \upsilon_L)}{K(\eta_L, \upsilon_L) + K(\eta_R, \upsilon_R)} \tag{5}$$

Combining both distributions presented in Equations (2) and (4), the following density for z can be considered:

$$p(z|\eta_L, \upsilon_L, \eta_R, \upsilon_R, \alpha) = H_{LR} \cdot \begin{cases} \left(1 + \frac{1}{\eta_L}\left|\frac{z}{2\alpha}\right|^{\upsilon_L}\right)^{-\frac{\eta_L+1}{\upsilon_L}}, z \leq 0 \\ \left(1 + \frac{1}{\eta_R}\left|\frac{z}{2(1-\alpha)}\right|^{\upsilon_R}\right)^{-\frac{\eta_R+1}{\upsilon_R}}, z > 0 \end{cases} \tag{6}$$

where the following formula gives the normalising constant HLR:

$$H_{LR} = \frac{1}{\frac{\alpha}{K(\eta_L, \upsilon_L)} + \frac{1-\alpha}{K(\eta_R, \upsilon_R)}}$$

The density function in Equation (6) has a unit scale parameter and a zero mode. The probability that a random variable z is not positive is defined as follows (Harvey and Lange [34]):

$$P(z \leq 0) = \frac{\frac{\alpha}{K(\eta_L, \upsilon_L)}}{\frac{\alpha}{K(\eta_L, \upsilon_L)} + \frac{1-\alpha}{K(\eta_R, \upsilon_R)}} \tag{7}$$

3. Empirical insights into CAPM regression

The most important outcome from the CAPM, namely the β, is typically estimated by regressing the returns of individual stocks, portfolios of instruments, or branch indices on the overall market index, both adjusted for the risk-free return proxy. This is done according to the following regression model:

$$r_t - r_t^f = \beta(r_t^m - r_t^f) + \sqrt{\sigma^2} z_t, t = 1, \ldots, T \tag{8}$$

where r_t denotes the return on an asset (it can also be a branch sub-index or a portfolio), r_t^m denotes the return on the market portfolio, and r_t^f denotes the return on a risk-free asset. Additionally, t refers to the time the observations were taken, $\sqrt{\sigma^2}$ is the scale parameter of the error term, and z_t is a sequence of i.i.d. random variables with zero location and unit scale.

The model framework presented by Equation (8) has been widely used in empirical papers and has influenced other methods such as Fama-MacBeth

regressions. The sequence of z_t can be assumed to follow a standardised Gaussian distribution, where σ^2 refers to the variance. Furthermore, Equation (8) can be improved by adding an intercept in order to evaluate standard statistical tests for the CAPM theory. The CAPM theory, which imposes a zero restriction on the intercept in the null hypothesis, has been studied for decades, as detailed by Jensen et al. [45]. It's worth noting that the assumptions behind the CAPM theory do not require returns to be normally distributed, as stated by Adcock [24].

Chamberlain [46] pointed out that the mean-variance optimisation criterion, the basis of portfolio optimisation and CAPM, is consistent with an investor's portfolio decision-making only if the returns are elliptically distributed. Empirical asset pricing tests proposed by Gibbons et al. [47] are valid only under the normality assumption, which is a special case of elliptical distributions. Consequently, studies have been conducted to develop tests for cases where the distribution of returns is generally elliptical, including work by Affleck-Graves and McDonald [27], MacKinlay and Richardson [28], and Zhou [29].

However, the empirical distributions of financial returns do not fit the Gaussian case and, in many instances, the entire elliptical class. As a result, it is possible to relax the assumptions regarding *zt* and use richer parametrisation and more flexible sampling distributions for $r_t - r_t^f$ in Equation (8) that allow for asymmetry and heavy tails. This research direction has recently gained significant attention and is a viable alternative to estimating the conditional β using the M-GARCH framework or general multivariate volatility models.

Numerous attempts have been made to explicitly define the distribution of financial returns. Harvey and Siddique [30] stressed the importance of integrating distributional asymmetry into the asset pricing model, emphasizing the economic significance of systematic skewness and its influence on risk premium. Adcock [24] examined the ramifications of assuming the Azzalini skew-normal distribution of returns in the CAPM model. Li et al. [31], Bao et al. [33], and others modelled the distribution of the error term in Equation (8) using asymmetric power distributions or asymmetric exponential power distributions. Theodossiou and Theodossiou [32] analyzed the sensitivity of β parameter estimates in light of outliers in the series of stock market returns, revealing substantial bias in OLS estimates in the case of non-normal empirical distributions of financial returns. While Theodossiou and Theodossiou's [32] feasible estimation procedure does not explicitly define the distribution of the error term in CAPM regressions, it is equivalent to the OLS procedure (under the Gaussian model) and, importantly, yields significantly different β estimates in the case of heavy-tailed and asymmetric data. The pursuit of reliable inference in linear regression models with heavy-tailed errors has a longstanding tradition in the empirical sciences and has been of particular interest in financial economics for decades. Shephard [48] has recently detailed consistent and asymptotically normal estimators of regression parameters in the context of heavy-tailed predictors with heteroscedastic outcomes. The empirical illustration demonstrating the accuracy and utility of Shephard's [49] method mainly focuses on the estimation of β coefficients within CAPM regressions.

4. Empirical results—Selected branch indices from WSE

4.1. Data description and model specification

In the empirical part of the paper, we analyse a set of selected sub-indices listed on the Warsaw Stock Exchange (WSE). The examined time series concern the indices of the construction (WIG-Budow), IT (WIG-Info), media (WIG-Media), real estate (WIG-Nrchom), fuel (WIG-Paliwa), food (WIG-Spozyw), and telecommunications sectors (WIG-Telekom). The time range of the data covers observations from 3 January 2006 to 29 May 2020. In empirical research, we analyse daily, weekly, monthly, and quarterly data, giving 3600 daily, 751 weekly, 172 monthly, and 56 quarterly observations, respectively. WIG index quotations were used to approximate the market portfolio, and the risk-free rate was based on the yields of the Poland 10Y government bonds.

We have estimated the parameter β in regression Equation (8) using stochastic assumptions for z_t as given by Equation (6). Our general model utilises the generalised asymmetric student-*t* distribution proposed by Harvey and Lange [35], and we refer to this specification as MFullHL. We have also estimated the parameter β using two special cases regarding the distribution of z_t in Equation (8). The first one, MSymHL, is based on the distribution of z_t given by Equation (1). It is a special case of MHL resulting from imposing three restrictions: $\alpha = 0.5$, $\eta L = \eta R = \eta$, and $\upsilon L = \upsilon R = \upsilon$ in Equation (6). The second special case of MHL, denoted by MG, is the limiting one, referring to the case of the Gaussian distribution of z_t. We can obtain it through the following set of restrictions: $\alpha = 0.5$, $\upsilon L = 2$, $\upsilon R = 2$, $\eta L \to +\infty$, and $\eta R \to +\infty$.

The models MG and MSymHL ensure that the expected value of the error term, denoted ε_t, equals zero (if it exists in the latter case). However, the model MHL is different in that possible skewness or asymmetry (or both) may distort the expectation of ε_t from having a zero modal value in Equation (8). As a result, the model Equation (8) complemented by an intercept is not identifiable. This implies that from a statistical perspective, there is no difference between an intercept and a non-zero expected value of ε_t. Therefore, the researcher must decide whether to include an intercept in Equation (8) and ensure that $E(\varepsilon_t) = 0$, or not include an intercept and find observational opportunities that would allow for $E(\varepsilon_t) \neq 0$. The latter alternative can be exploited within the environments given by models that allow for skewness and/or asymmetry. Furthermore, within these models, it is possible to perform simple likelihood ratio tests of the Capital Asset Pricing Model (CAPM) theory, which posits that $E(\varepsilon_t) = 0$, against the alternative hypothesis of skewness, asymmetry, or both.

4.2. Empirical analyses

We have presented the results of the Maximu Likelihood (ML) estimation of parameters in **Table 1**. We have also reported the values of P ($z_t < 0$) obtained considering all competing sampling distributions when analysing daily data. Additionally, we have calculated the log-likelihood values at the ML estimates. We want to highlight that the estimates of the β parameter are sensitive to the choice of assumptions regarding the error term. However, we expected differences to be much stronger than those reported. The analyses suggest that, except for WIG-Paliwa, β is

estimated to be larger in the case of the Gaussian error term (MG), compared to MSymHL or MHL. However, the risk assessment based on β is almost the same for both non-Gaussian specifications, as the β parameter is estimated to be almost the same value. Estimating parameter σ^2 differs among models due to different interpretations, since it is the variance only in the case of MG. The inference about shape parameters is uncertain in all analysed sub-indices. The data clearly indicates that the error term departs from the Gaussian case. However, there is strong uncertainty about parameters *v* and η. The data does not strongly support the two-piece mechanism enforcing asymmetry in model MHL. Although we report differences between *vL* and *vR* or ηL and ηR, the range of uncertainty determined by the standard errors suggests that these differences may not be statistically significant. Specifically, for the WIG-Info subindex, according to model MHL, the tail behaviour is definitely of a different nature on the left ($\eta\hat{L} \approx 8.83$) than on the right ($\eta\hat{R} \approx 17.47$). However, the standard errors, reaching 4.32 in the case of parameter ηR, indicate that a symmetric tails case is also supported. Additionally, shape parameters *vL* and *vR* are estimated to be less than 2 ($\eta^* = 2$ indicates student-*t* shape), but the huge standard errors disable precise inference about this feature of the distribution of the error term. The same results are obtained for the WIG-Paliwa index.

The *p*-values of the likelihood ratio (LR) tests, which are shown in **Table 1**, indicate that the Gaussian error term is not supported by the data. The null hypothesis is rejected for both MSymHL and MHL alternatives. However, differences between MSymHL and MHL are unclear. Only in the case of WIG-Nrchom and WIG-Media does the unconstrained model MHL receive stronger data support, and the restriction to the symmetric case (MSymHL) is rejected at a reasonable level of statistical significance. The *p*-values of the appropriate LR test are 0.0034 and 0.0255 for WIG-Nrchom and WIG-Media, respectively. All remaining time series do not support the asymmetry obtained by the two-piece mechanism.

There is another source of asymmetry in MHL, referring to the inverse scale factors mechanism, with skewness measure $\alpha \in (0, 1)$; see Equation (2). Empirical analyses conducted here yield a little data support against $\alpha = 0.5$ (assuring symmetry in Equation (2)). With regard to the point estimates of the α parameter, it can be stated that WIG-Budow, WIG-Info, WIG-Media and WIG-Telekom indicate negative skewness ($\hat{\alpha} < 0.5$), while WIG-Nrchom, WIG-Paliwa and WIG-Spozyw support positive skewness, but in case of the latter one, very weak.

The mechanisms of the two-pierce and inverse scale factors are different in nature. Hence, the data may support different directions of asymmetry generated by both mechanisms. In the case of the two-piece mechanism, a heavier left tail indicates asymmetry to the left of z_t in Equation (8), while a heavier right tail—indicates asymmetry to the right. A quantity of interest that summarises information of the scale of departure of the distribution of the error term from the symmetric case is the $P(z_t < 0)$; see Equation (7) for MHL. Models MG and MSymHL are built on the basis of the error term with symmetric distribution, and hence they assure that $P(z_t < 0) = P(z_t \geq 0) = 0.5$. Consequently, formal statistical inference about $P(z_t < 0)$ is the possible only condition to the model MHL. The point estimates and standard errors of this function of interest are presented in **Table 1**. Except for WIG-Info, analysed sub-indices

support negative asymmetry of the error term in CAPM regression. As described previously, the results of inference about the shape and tails of the distribution of z_t in Equation (8) were quite different when comparing estimation outcomes from different datasets. Great uncertainty about η and ν was in common. The estimation of $P(z_t < 0)$ seems to be characterised by the same level of statistical uncertainty in each analysed time series. The deviation from the symmetric case is not big but precisely estimated since the approximated standard error is not greater than 0.0046.

The empirical analysis presented above was replicated for data on weekly, monthly, and quarterly frequency for each analysed sub-index. The resulting outcome is summarised in **Figures 1** and **2**, which show the estimated CAPM lines against the data points for WIG-Nrchom and WIG-Paliwa, respectively. The complete set of estimated CAPM lines is presented in Appendix in **Figures A1–A3**. The black line represents a hypothetical CAPM with the slope $\beta = 1$. The blue line corresponds to the estimated β under MG, while the red line is the case of MHL, which is almost the same as in the case of model MSymHL. WIG-Nrchom and WIG-Paliwa were chosen to demonstrate the sensitivity of the inference about β with respect to the frequency of the analysed time series.

Table 1. ML estimates and asymptotic standard errors (in brackets) of parameters and of $P(z_t < 0)$, natural logarithm of the ML likelihood values with the results of the LR test; the case of the daily data.

Index	Model	β	σ^2	ηL	υL	ηR	υR	α	$P(z_t < 0)$	Loglik	p-value of LR test	
											MG	MSymHL
	MG	0.7408 (0.0136)	1.0508 (0.0004)	+∞	2	+∞	2	0.5	0.5	−5150.716		
WIG-Budow	MSymHL	0.7077 (0.0326)	0.6469 (0.2002)	6.91 (0.48)	1.71 (1.02)	ηL	υL	0.5	0.5	−5068.580	$p < 10{-}20$	
	MHL	0.6998 (0.0560)	0.6474 (0.3155)	5.78 (1.06)	1.89 (3.01)	8.70 (4.87)	1.55 (0.24)	0.4984 (0.0713)	0.5130 (0.0044)	−5066.44	$p < 10{-}20$	0.1177
	MG	0.6751 (0.0129)	0.9538 (0.000374)	+∞	2	+∞	2	0.5	0.5	−4994.399		
WIG-Info	MSymHL	0.6566 (0.0449)	0.6559 (0.3306)	11.01 (0.98)	1.64 (1.52)	ηL	υL	0.5	0.5	−4944.953	$p < 10{-}20$	
	MHL	0.6569 (0.0149)	0.6580 (0.5375)	8.03 (1.03)	1.79 (2.10)	17.47 (4.32)	1.52 (5.19)	0.4869 (0.0542)	0.4995 (0.0043)	−4943.265	$p < 10{-}20$	0.1849
	MG	0.7268 (0.0159)	1.4507 (0.000570)	+∞	2	+∞	2	0.5	0.5	−5710.793		
WIG-Media	MSymHL	0.7111 (0.0217)	0.9420 (0.1099)	4.89 (1.89)	2.23 (0.09)	ηL	υL	0.5	0.5	−5633.865	$p < 10{-}20$	
	MHL	0.7125 (0.0159)	0.9517 (0.1842)	3.86 (3.82)	2.55 (0.17)	8.47 (3.99)	1.86 (0.36)	0.4835 (0.0363)	0.5047 (0.0042)	−5630.197	$p < 10{-}20$	0.0255
	MG	0.8617 (0.0138)	1.0862 (0.000427)	+∞	2	+∞	2	0.5	0.5	−5152.197		
WIG-Nrchom	MSymHL	0.8244 (0.1070)	0.4692 (0.3888)	5.30 (0.15)	1.42 (2.37)	ηL	υL	0.5	0.5	−4906.878	$p < 10{-}20$	
	MHL	0.8213 (0.0672)	0.4652 (0.2757)	6.87 (15.83)	1.33 (0.18)	3.75 (0.27)	1.64 (0.59)	0.5273 (0.0626)	0.5038 (0.0046)	−4901.192	$p < 10{-}20$	0.0034

Table 1. (*Continued*).

Index	Model	β	σ^2	ηL	vL	ηR	vR	α	$P(z_t < 0)$	Loglik	p-value of LR test	
WIG-Paliwa	MG	1.0841 (0.0153)	1.3379 (0.000526)	+∞	2	+∞	2	0.5	0.5	−5590.365		
	MSymHL	1.1061 (0.1051)	0.8803 (0.2611)	8.14 (0.64)	1.73 (0.55)	ηL	vL	0.5	0.5	−5526.729	$p < 10{-}20$	
	MHL	1.1074 (0.0147)	0.8855 (0.0330)	14.86 (2.00)	1.61 (1.23)	6.48 (2.84)	1.76 (2.28)	0.5103 (0.0333)	0.5013 (0.0044)	−5525.163	$p < 10{-}20$	0.2089
WIG-Spozyw	MG	0.6334 (0.0161)	1.4781 (0.000581)	+∞	2	+∞	2	0.5	0.5	−5736.596		
	MSymHL	0.5765 (0.0365)	0.8718 (0.2092)	4.86 (0.43)	2.00 (0.89)	ηL	vL	0.5	0.5	−5629.440	$p < 10{-}20$	
	MHL	0.5774 (0.0582)	0.8711 (0.1165)	4.77 (6.97)	2.13 (0.27)	5.18 (8.44)	1.85 (1.35)	0.5012 (0.2077)	0.5089 (0.0044)	−5627.984	$p < 10{-}20$	0.2332
WIG-Telekom	MG	0.6752 (0.0181)	1.8720 (0.000735)	+∞	2	+∞	2	0.5	0.5	−6014.624		
	MSymHL	0.6536 (0.0480)	1.0100 (0.2053)	4.03 (1.44)	2.40 (1.66)	ηL	vL	0.5	0.5	−5849.315	$p < 10{-}20$	
	MHL	0.6547 (0.0472)	1.0123 (0.0628)	3.70 (2.99)	2.67 (0.19)	4.68 (2.89)	2.10 (0.28)	0.4997 (0.0424)	0.5118 (0.0043)	−5847.148	$p < 10{-}20$	0.1145

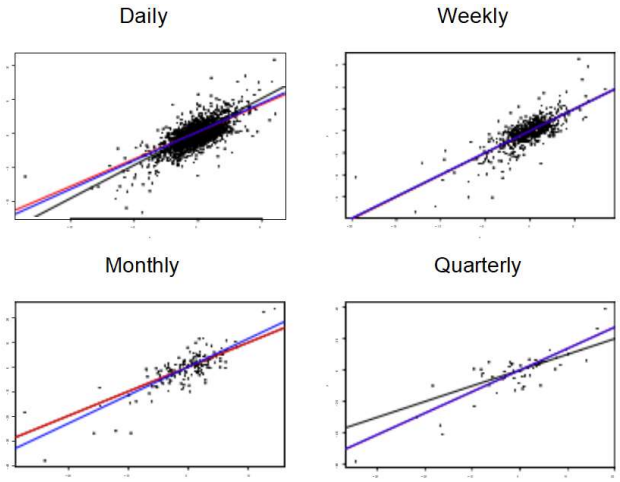


Figure 1. Estimated CAPM lines for WIG-Nrchom against the data of different frequency.

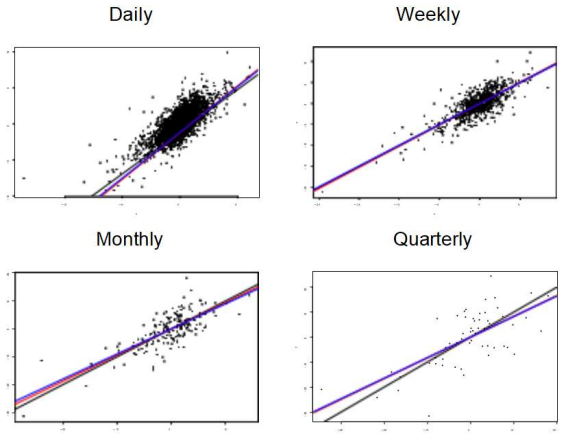


Figure 2. Estimated CAPM lines for WIG-Paliwa against the data of different frequency.

In the case of WIG-Nrchom, depicted in **Figure 1**, the slope of the CAPM line becomes steeper as the frequency of the data decreases. Daily data suggests that $\beta < 1$ for both models, but estimation based on the weekly time series results in $\hat{\beta}$ that is almost equal to one. In the case of monthly data, inference about β differs across models, where model MG yields $\hat{\beta} \approx 1.1560$ (with the standard error 0.0609) and model MHL gives $\hat{\beta} \approx 1.0000$ (with the standard error 0.000885). The quarterly data set supports $\beta < 1$ for both models.

The results of analysis of the second case (WIG-Paliwa) are summarised in **Figure 2**. The point estimates of the β are relatively similar in model MG and MHL for each analysed frequency of the data. However, they decrease as the frequency of data decreases. In particular, the daily data suggests that $\beta > 1$, and the weekly frequency gives a result where β is almost equal to one in both models. The monthly and quarterly series indicate that $\beta < 1$.

5. Conclusion

In this paper, we analysed CAPM regression under a class of distributions allowing for various exceptions from the Normal family. Namely, we assume the error

term belongs to the generalised asymmetric student-*t* class that Harvey and Lange [34] elaborated, allowing for skewness and asymmetric tail thickness. The Harvey and Lange [34] generalisation unifies the student-*t* family of distributions with the GED (Generalised Error Distributions) class in an elegant parametrisation. We focused on the Warsaw Stock Exchange (WSE) indices and analysed estimates of the β parameter for a set of selected branch sub-indices.

For most analysed time series, the data suggests that β is estimated to be larger in the case of the Gaussian error term compared to models with error terms allowing for heavy tails and asymmetry. However, the risk assessment based on β is almost the same for all non-Gaussian specifications, as the β parameter is estimated to be nearly the same value. The data does not strongly support the two-piece mechanism enforcing asymmetry in the distribution of the error term. Only in the case of WIG-Nrchom and WIG-Media does the unconstrained model receive more vital data support, and the restriction to the symmetric distribution is rejected at a reasonable level of statistical significance. All remaining time series do not support the asymmetry obtained by the two-piece mechanism.

In the paper, we also demonstrated how risk assessment—elaborated on the basis of estimation of the β parameter—may change with respect to the frequency of analysed time series. The main conclusion that arises from our research is that estimation of the β parameter may change across models with different assumptions imposed on the distribution of the error term and, more importantly, across different data frequencies. Consequently, the risk assessment resulting from the estimated CAPM model may not correspond to the analysed financial instrument only. The frequency at which the researcher observes the analysed time series plays as important a role as the original choice of the instrument.

Funding: This research was supported by the grant number 59/GAE/2020/POT financed by Cracow University of Economics, Poland. Some previous attempts to analyse CAPM were performed by Gomola and Pipień [50].

Conflict of interest: The author declares no conflict of interest.

References

1. De Forest E. On an Unsymmetrical Probability Curve. Analyst. 1882; 9: 135-142.
2. De Forest E. On an Unsymmetrical Probability Curve. Analyst. 1883; 10: 1-7.
3. Edgeworth FY. The law of error and the elimination of chance. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science. 1886; 21(131): 308-324. doi: 10.1080/14786448608627852
4. Pearson K. Asymmetrical Frequency Curves. Nature. 1893; 48: 615-616.
5. Pearson K. On Skew Probability Curves. Nature. 1895; 52: 317.
6. Genton MG. Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality. Chapman & Hall/CRC, Boca Raton; 2004.
7. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973; 81: 637-659.
8. Merton RC. Theory of Rational Option Pricing. Bell Journal of Economics. 1973; 4: 141-183.
9. Markovitz HM. Portfolio Selection. Journal of Finance. 1952; 7: 77-91.
10. Sharpe WF. Capital asset prices: a theory of market equilibrium under conditions of risk. The Journal of Finance. 1964; 19(3): 425-442. doi: 10.1111/j.1540-6261.1964.tb02865.x
11. Treynor JL. Toward a Theory of Market Value of Risky Assets. 1961; Unpublished Work.

12. Lintner J. Security Prices, Risk, and Maximal Gains from Diversification. The Journal of Finance. 1965; 20(4): 587. doi: 10.2307/2977249
13. Lintner J. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. Review of Economics and Statistics. 1965; 47: 13-37.
14. Graham JR, Harvey CR. The Theory and Practice of Corporate Finance: Evidence from the Field. Journal of Financial Economics. 2001; 60: 187-243.
15. Altavilla C, Bochmann P, Ryck JD, et al. Measuring the Cost of Equity of Euro Area Banks. ECB Occasional. 2021. doi: 10.2139/ssrn.3797125
16. Kaplan PD, Peterson JD. Full-Information Industry Betas. Financial Management. 1998; 27(2): 85-93.
17. Fama EF, French KR. The Capital Asset Pricing Model: Theory and Evidence. Journal of Economic Perspectives. 2004; 18: 25-46.
18. Andrei D, Cujean J, Wilson M. The Lost Capital Asset Pricing Model, The Review of Economic Studies. 2023; 90(6): 2703-2762. doi: 10.1093/restud/rdad013
19. Savor P, Wilson M. Asset pricing: A tale of two days. Journal of Financial Economics. 2014; 113(2): 171-201. doi: 10.1016/j.jfineco.2014.04.005
20. Black F. Capital Market Equilibrium with Restricted Borrowing. Journal of Business. 1972; 45(3): 444-455. doi: 10.1086/295472 doi
21. Fama EF, French K. Value Versus Growth: The International Evidence. Journal of Finance. 1998; 53(6): 1975-1999. doi: 10.1111/0022-1082.00080
22. Zhou XY, Yin G. Markowitz's Mean-Variance Portfolio Selection with Regime Switching: A Continuous-Time Model. SIAM Journal on Control and Optimization. 2003; 42(4): 1466-1482. doi: 10.1137/s0363012902405583
23. Wei L. Empirical Study of CAPM in the 5G sector of Chinese stock market. In: Proceedings of the 2021 2nd International Conference on Big Data Economy and Information Management (BDEIM); 3-5 December 2021; Sanya, China.
24. Urom C, Chevallier J, Zhu B. A dynamic conditional regime-switching GARCH CAPM for energy and financial markets. Energy Economics. 2020; 85: 104577. doi: 10.1016/j.eneco.2019.104577
25. Adcock C. Capital Asset Pricing for UK Stocks under the Multivariate Skew-Normal Distribution. In: Genton MG (editor). Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality. Chapman & Hall/CRC, Boca Raton; 2004.
26. Azzalini A, Dalla Valle A. The Multivariate Skew-Normal Distribution. Biometrika. 1996; 83: 715-726.
27. Adcock CJ, Shutes K. Fat Tails and the Capital Asset Pricing Model. In: Dunis CL (editor). Advances in Quantitative Asset Management. Kluwer Academic Publishers; 2000.
28. Affleck-Graves J, McDonald B. Nonnormalities and tests of asset-pricing theories. Journal of Finance. 1989; 44: 889-908.
29. MacKinlay AC, Richardson MP. Using generalized method of moments to test mean variance efficiency. Journal of Finance. 1991; 46: 511-527.
30. Zhou G. Asset-Pricing Tests Under Alternative Distributions. The Journal of Finance. 1993; 48(5): 1927. doi: 10.2307/2329073
31. Harvey CR, Siddique A. Conditional Skewness in Asset Pricing Tests. Journal of Finance. 2000; 55: 1263-1295. doi: 10.1111/0022-1082.00247
32. Li L, Gan Q, Zhuo Z, et al. Testing the CAPM Theory Based on a New Model for Fama-French 25 Portfolio Returns. Theoretical Economics Letters. 2014; 04(08): 666-680. doi: 10.4236/tel.2014.48085
33. Theodossiou AK, Theodossiou P. Stock return outliers and beta estimation: The case of U.S. pharmaceutical companies. Journal of International Financial Markets, Institutions and Money. 2014; 30: 153-171. doi: 10.1016/j.intfin.2014.02.002
34. Bao T, Diks C, Li H. A generalized CAPM model with asymmetric power distributed errors with an application to portfolio construction. Economic Modelling. 2018; 68: 611-621. doi: 10.1016/j.econmod.2017.03.035
35. Harvey A, Lange RJ. Volatility Modeling with a Generalized t-Distribution. Journal of Time Series Analysis. 2017; 38(2): 175-190. doi: 10.1111/jtsa.12224
36. Li R, Nadarajah S. A review of Student's t distribution and its generalizations. Empirical Economics. 2018; 58: 1461-1490. doi: 10.1007/s00181-018-1570-0
37. Zhu D, Zinde-Walsh V. Properties and estimation of asymmetric exponential power distribution. Journal of Econometrics. 2009; 148(1): 86-99. doi: 10.1016/j.jeconom.2008.09.038

38. Zhu D, Galbraith JW. A generalized asymmetric Student-t distribution with application to financial econometrics. 2010; Journal of Econometrics. 2010; 157(2): 297-305. doi: 10.1016/j.jeconom.2010.01.013
39. Fernández C, Steel MF. On Bayesian modeling of fat tails and skewness. Journal of the American Statistical Association. 1998; 93(441): 359-371. doi: 10.1080/01621459.1998.10474117
40. Theodossiou P. Financial data and the skewed generalized t distribution. Management Science. 1998; 44: 1650-1661.
41. Mazur B, Pipień M. Modelling Time Varying Asymmetry and Tail Behaviour in Long Series of Daily Financial Returns. Studies in Nonlinear Dynamics and Econometrics. 2018; 22(5).
42. Mazur B, Pipień M. A Family of Flexible Multivariate Distributions with Applications in Empirical Finance. Acta Physica Polonica A. 2020; 138: 65-73. doi: 10.12693/APhysPolA.138.65
43. Ferreira JTAS, Steel MFJ. A Constructive Representation of Univariate Skewed Distributions. Journal of the American Statistical Association. 2006; 101: 823-829.
44. Hansen B. Autoregressive Conditional Density Estimation. International Economic Review. 1994; 35(3): 705-730.
45. Arnold BC, Groeneveld RA. Measuring the Skewness with respect to the Mode. The American Statistician. 1995; 49(1): 34-38.
46. Jensen MC, Black F, Scholes MS. The Capital Asset Pricing Model: Some Empirical Tests. In: Jensen MC (editor). Studies in the Theory of Capital Markets. Praeger Publishers Inc.; 1972.
47. Chamberlain G. A characterization of the distributions that imply mean-variance utility functions. Journal of Economic Theory. 1983; 29: 185-201.
48. Gibbons MR, Ross SA, Shanken J. A Test of the Efficiency of a Given Portfolio. Econometrica. 1989; 57(5): 1121. doi: 10.2307/1913625
49. Shephard N. An Estimator for Predictive Regression: Reliable Inference for Financial Economics, manuscript. Available online: https://arxiv.org/abs/2008.06130 (accessed on 27 January 2024).
50. Gomola A, Pipień M. Maximum Likelihood Estimates of CAPM Beta's for the Banking Sector Indices under Heavy Tailed and Asymmetric Distributions - the Case of NYSE and WSE. 2022; Unpublished Work.

Appendix

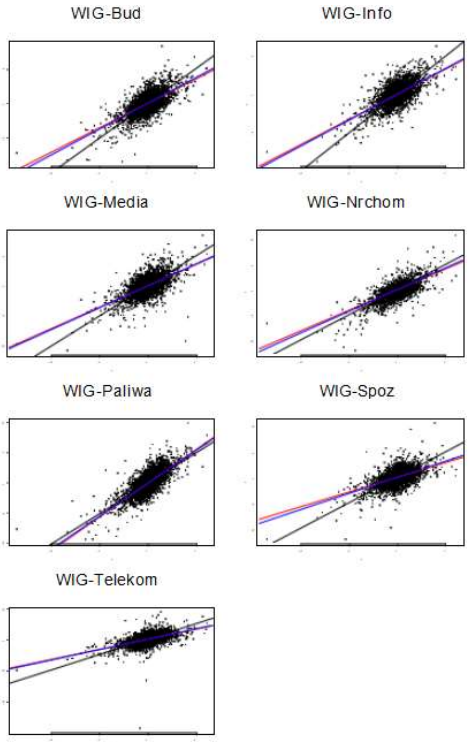


Figure A1. Estimated CAPM lines, the case of daily data.

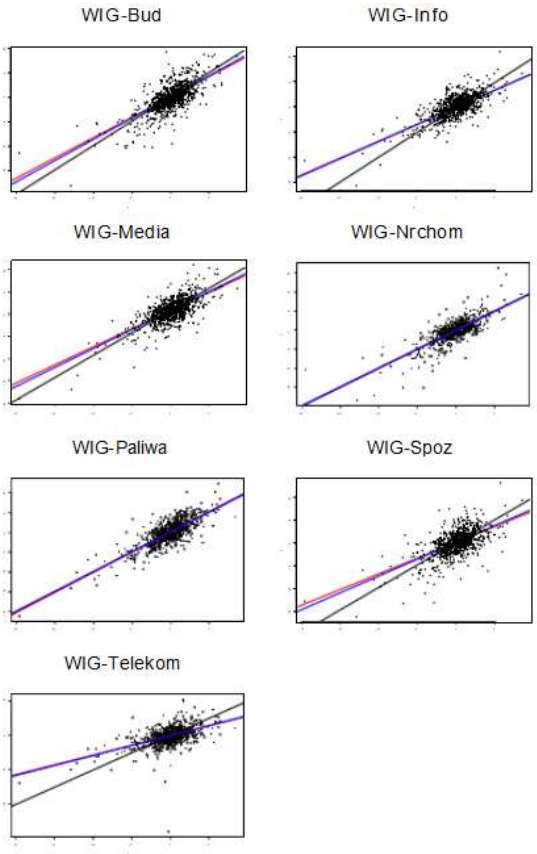


Figure A2. Estimated CAPM lines, the case of weekly data.

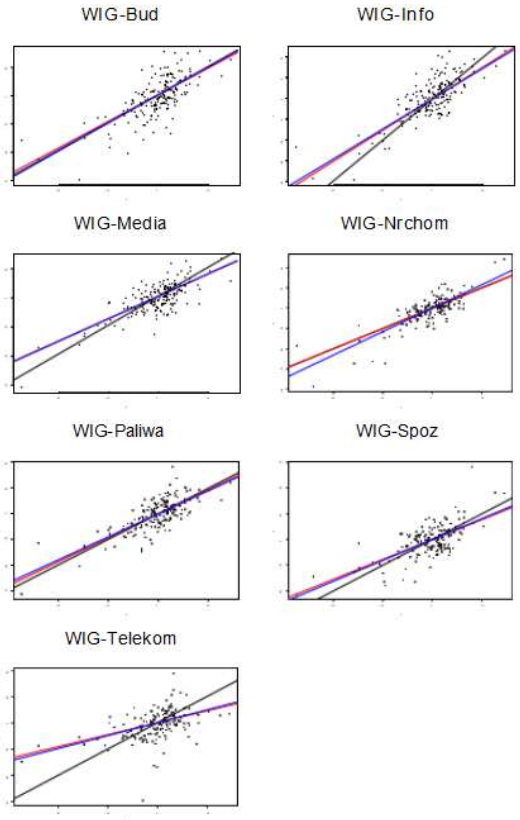


Figure A3. Estimated CAPM lines, the case of monthly data.

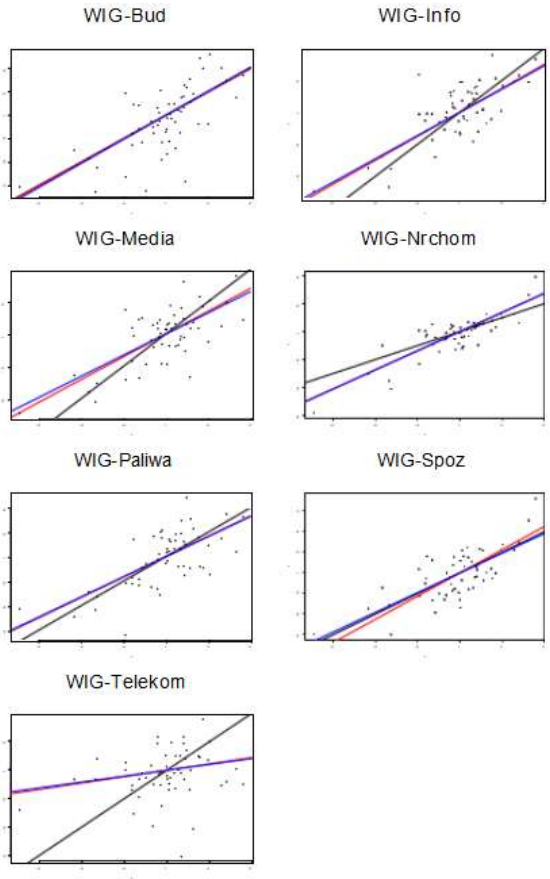


Figure A4. Estimated CAPM lines, the case of quarterly data.