Characterization and Application of Nanomaterials (2023) Volume 6 Issue 1
doi:10.24294/can.v6i1.1953

ORIGINAL RESEARCH ARTICLE

Is creating materials with a desired refraction coefficient practically possible?

Alexander G. Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602, USA. E-mail: ramm@ksu.edu

ABSTRACT

A theory of many-body wave scattering is developed under the assumption $a << d << \lambda$, where a is the characteristic size of the small body, d is the distance between neighboring bodies and λ is the wave-length in the medium in which the bodies are embedded. The multiple scattering is essential under these assumptions. The author's theory is used for creating materials with a desired refraction coefficient. This theory can be used in practice. A recipe for creating materials with a desired refraction coefficient is formulated. Materials with a desired radiation pattern, for example, wave-focusing materials, can be created.

PACS: 02.30.Rz; 02.30.Mv; 41.20.Jb

MSC: 35Q60; 78A40; 78A45; 78A48

Keywords: Wave Scattering by Many Small Bodies; Smart Materials

ARTICLE INFO

Received: 8 February 2023
Accepted: 2 March 2023
Available online: 16 March 2023

1. Introduction

The aim of this paper is to give an affirmative answer to the question in the title of this paper. This brings potentially many possibilities for progress in technology.

There is a large literature on wave scattering by small bodies, starting from Rayleigh's work (1871)[1–3]. If the scatterer is small then the scattered field can be calculated analytically for bodies of arbitrary shapes, see reference [4].

The many-body wave scattering problem was discussed in the literature mostly numerically, if the number of scatterers was small, or under the assumption that the influence of the waves, scattered by other particles on a particular particle is negligible[5]. This corresponds to the case when the distance d between neighbouring particles is much larger than the wavelength λ, and the characteristic size a of a small body (particle) is much smaller than λ. Theoretically and practically the assumptions

$$a << \lambda, \quad d >> \lambda, \tag{1}$$

are the simplest ones which allow one to neglect multiple scattering. By $k = \frac{2\pi}{\lambda}$, the wave number is denoted.

In the author's theory, the basic assumptions are

$$a << d << \lambda, \tag{2}$$

and the multiple scattering is of basic importance under these assumptions[4,6–35]. It is clear that assumption (2) can be practically realized. Its importance comes from the fact that the author gave a rigorous asymptotically exact solution of the many-body scattering problem under assumption (2) when $a \to 0$. This solution can be well approximated numerically by the particles of the size $a > 30$ nm. Practically the size of a can be found by comparison of the solution for some a and for $\frac{a}{2}$. If these solutions are practically close, then one considers this a as suitable. The aim of this paper is to show that our theory can be used practically.

In reference [36], for the first time the author's theory was used for solving the scattering problem for 10 billion small particles. This problem was solved numerically and numerical results were presented.

Let us formulate the wave scattering problems we deal with. Let D be a bounded domain in $\mathbb{R}^3$ with a sufficiently smooth boundary. The scattering problem consists of finding the solution to the problem:

$$(\nabla^2 + k^2)u = 0 \ \text{ in } G' := \mathbb{R}^3 \backslash G,\ G := \cup_{m=1}^{M} D_m, \quad k = const > 0, \tag{3}$$

where $D_m = B(x_m, a)$ is an impedance ball, centered at x_m and of small radius a,

$$u = u_0 + v,\ u_0 = e^{ik\alpha\cdot x},\ \alpha \in S^2, \tag{4}$$

S^2 is the unit sphere in $\mathbb{R}^3$, u_0 is the incident field, v is the scattered field satisfying the radiation condition

$$v_r - ikv = o(\tfrac{1}{r}),\ r := |x| \to \infty,\ v_r := \frac{\partial v}{\partial r}, \tag{5}$$

and u satisfies the impedance boundary condition (bc) on the boundary of G:

$$u_N - \zeta_m u = 0, \quad \text{on } S_m, \quad \mathrm{Im}\zeta_m \le 0, \tag{6}$$

where ζ_m is a constant, N is the unit normal to $S := \cup_{m=1}^{M} S_m$, pointing out of $G := \cup_{m=1}^{M} D_m$, and S_m is the surface of $D_m = B(x_m, a)$.

By refraction coefficient $n(x)$ the coefficient in the equation

$$(\nabla^2 + k^2 n^2(x))u = (\nabla^2 + k^2 - q(x))u = 0 \tag{7}$$

is understood, where $q(x) := k^2(n^2(x) - 1)$.

Let $g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$. Then $(\nabla^2 + k^2)g(x, y) = -\delta(x - y)$, where $\delta(x)$ is the delta function.

Let us distribute small impedance particles $D_m = B(x_m, a)$ in D so that

$$\mathbb{N}(\Delta) = a^{\kappa-2}|\Delta|[1 + o(1)],\ a \to 0, \tag{8}$$

where $\Delta \subset D$ is an arbitrary connected open subset of D, $|\Delta|$ is its volume, $\kappa \in (0, 1)$ is a number the experimenter may choose arbitrarily and $\mathbb{N}(\Delta)$ is the number of particles in Δ. Throughout this paper the important assumptions $a << d << \lambda$ and (8) are satisfied. As $a \to 0$, the number of small particles $\mathbb{N}(\Delta)$ in (8) tends to infinity since $\kappa - 2 < 0$.

We assume in this paper (for simplicity only) that the small particles are distributed in the domain D and the refraction coefficient in D equals to 1. In the monograph [31], it is assumed that D is filled with the material whose refraction coefficient $n_0(x)$ is known and we wanted to create in D the material with the desired refraction coefficient $n(x)$.

The boundary impedances ζ_m are chosen by the formula

$$\zeta_m = a^{-\kappa}h(x_m), \tag{9}$$

where $h(x)$ is a continuous function in D, $\mathrm{Im}h \le 0$.

It will be clear from Section 3 that the function $h(x)$ can be determined by choosing a suitable boundary impedance $\zeta(x)$. When $a \to 0$, the ζ_m and $h(x_m)$ can be considered as continuous functions $\zeta(x)$ and $h(x)$.

2. Solution of many-body scattering problem

We look for the solution of the form

$$u = u_0 + \sum_{m=1}^{M} \int_{S_m} g(x, s)\sigma_m(s)ds = \sum_{m=1}^{M} g(x, x_m)Q_m + J, \tag{10}$$

where $\sigma_m(s)$ are unknown, $Q_m := \int_{S_m} \sigma_m(s)ds$. One may think about σ_m as of charge densities on S_m and of Q_m as of total charge on the surface S_m. We prove that

$$J := \sum_{m=1}^{M} \int_{S_m} [g(x,s) - g(x,x_m)]\sigma_m(s)ds \tag{11}$$

is negligible compared to

$$I := \sum_{m=1}^{M} g(x,x_m)Q_m, \tag{12}$$

so

$$J << I \text{ as } a \to 0. \tag{13}$$

We prove that the field u satisfies the following integral equation as $a \to 0$:

$$u(x) = u_0(x) - 4\pi\int_D g(x,y)h(y)u(y)dy, \tag{14}$$

where $h(x_m) = \frac{\zeta_m}{a^\kappa}$, and, since there are sufficiently many points $x_m \in D$, the function $h(x)$ is uniquely determined in D if the boundary impedances are known.

Apply the operator to $\nabla^2 + k^2$ to both sides of equation (14) and get

$$\left(\nabla^2 + k^2 - 4\pi h(x)\right)u(x) := \left(\nabla^2 + k^2 n^2(x)\right)u(x) = 0 \tag{15}$$

Therefore,

$$n^2(x) = 1 - 4\pi k^{-2}h(x). \tag{16}$$

We omit details since they can be found in the author's publications listed in the References, in particular, in monograph [31].

If originally in D were material with the known refraction coefficient $n_0(x)$, then formula (16) were $n^2(x) = n_0^2(x) - 4\pi h(x)N(x)k^{-2}$, where $N(x)$ is the distribution density for the small particles, see reference [31]. In this paper, we assume (for simplicity only) that $N(x) = 1$, see formula (8).

3. Recipe for creating materials with a desired refraction coefficient

Let us formulate a recipe for creating materials with a desired refraction coefficient. Formula (16) shows that if $h(x)$ is chosen properly, then any $n(x)$ can be obtained in D.

Recipe for creating materials with a desired refraction coefficient:

a) Calculate by formula (16) *the function $h(x)$;*

b) Distribute small impedance balls in the domain D by the distribution law (8). *The boundary impedances of these balls are defined by the function* $h(x)$.

Theorem 1. *The refraction coefficient of the resulting medium tends to the desired coefficient* $n(x)$ *as* $a \to 0$.

Let us show that practically negative refraction coefficient $n(x)$ can be obtained by the above recipe. Denote $b := 4\pi k^{-2} > 0$ and write equation (16) as

$$n(x) = (1 - bh(x))^{1/2} = |1 - bh(x)|^{1/2}e^{\phi/2}, \tag{17}$$

where ϕ is the argument of $1 - bh(x)$. Since the operator in (14) is of Fredholm type, it remains Fredholm type under small perturbations. Therefore one can take $h - i\epsilon$, where $\epsilon > 0$ is sufficiently small, and equation (14) will still have a unique solution.

By choosing h so that $\mathrm{Re}(1 - bh) > 0$ and $\mathrm{Im}(1 - bh) < 0$ and small, one gets the argument $\phi = 2\pi - \delta$, where $\delta > 0$ is arbitrarily small if ϵ is sufficiently small. Then $n(x)$ will be nearly negative: its argument will be $\pi - \delta/2$.

4. Creating materials with a desired radiation pattern

Let us define what we mean by the radiation pattern. Consider the scattering problem for the equation:

$$\nabla^2 u + k^2 u - q(x)u = 0,\ u = e^{ik\alpha\cdot x} + v, \tag{18}$$

where v satisfies the radiation condition. Assume that $k > 0$ and $\alpha \in S^2$ are fixed. Then the scattering amplitude $A(\beta, \alpha, k) = A(\beta)$, where the dependence on k, α is dropped since k and α are fixed. The formula for the scattering amplitude is known, see, e.g., reference [35]:

$$A(\beta) := A_q(\beta) = -\frac{1}{4\pi}\int e^{-ik\beta\cdot y}q(y)u(y)dy. \tag{19}$$

We call $A(\beta)$ the radiation pattern.

Consider an inverse problem (IP):

Given an arbitrary $f(\beta) \in L^2(S^2)$ *and an arbitrary small* $\epsilon > 0$, *can one find a* $q \in L^2(D)$ *such that*

$$\left\|f(\beta) - A_q(\beta)\right\|_{L^2(S^2)} < \epsilon. \tag{20}$$

This inverse problem was not formulated and was not studied in the works of other authors, to our knowledge.

Our result is stated in Theorem 2.

Theorem 2. *For any* $f(\beta) \in L^2(S^2)$ *and an arbitrary small* $\epsilon > 0$ *there is a* $q \in L^2(D)$ *such that* (20) *holds.*

Since small perturbations of q result in small perturbations of $A(\beta)$, there are infinitely many potentials q for which inequality (20) holds.

The conclusion of Theorem 2 follows from lemmas 3 and 4.

Lemma 3. The set $\left\{\int_D e^{-ik\beta\cdot x}h(x)dx\right\}_{\forall h\in L^2(D)}$ is dense in $L^2(S^2)$.

Corollary 1. Given $f \in L^2(S^2)$ and $\epsilon > 0$, one can find $h \in L^2(D)$ such that

$$\left\| f(\beta) + \frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx \right\| < \epsilon.$$

Lemma 4. The set $\{q(x)u(x,\alpha)\}_{\forall q\in L^2(D)}$ is dense in $L^2(D)$.

Corollary 2. Given $h \in L^2(D)$ and $\epsilon > 0$, one can find $q \in L^2(D)$ such that

$$\|h(x) - q(x)u(x,\alpha)\|_{L^2(D)} < \epsilon.$$

Since the scattering amplitude

$$A(\beta) = -\frac{1}{4\pi}\int_D e^{-ik\beta\cdot x}h(x)dx$$

depends continuously on h, the inverse problem **IP** is solved by Lemmas 3 and 4.

Proofs are omitted. They can be found in reference [31].

5. Discussion

How is the theory, outlined in the previous sections, can be used practically?

To create a material with a desired refraction coefficient, or a material with a refraction coefficient close to the desired, is practically very important. To my knowledge, there were no general methods for creating material with a desired refraction coefficient. To use the theory, outlined in this paper and in the monographs[31–33], one has to solve a technological problem: how to prepare a small particle, say, a ball of radius a, with the prescribed boundary impedance ζ. This problem should be solvable, see reference [33] for arguments supporting this conclusions. If this technological problem is solved, then the recipe outlined in this paper (and in the author's monographs[31–33] can be immediately used in practice.

The problem of creating materials with a desired radiation pattern, the wave focusing materials, for example, was not investigated earlier. This problem is of great practical interest. The usual bodies scatter waves mostly backwards, somewhat sidewise and a little forwards. If one creates a body which scatters waves, for example, in a given solid angle, this would be of great practical interest. Such a body can be created as follows from the theory outlined in the previous Section.

The author wrote this paper in an attempt to draw attention of the specialists in material sciences to the theory he has developed for creating materials with the desired refraction coefficient.

The author is not aware of the experimental results based on his theory. Such results are very desirable. There are numerical results, based on his theory, see references [37] and [38].

Conflict of interest

Author declares no conflict of interest.

References

1. Rayleigh J. Scientific papers. Cambridge: Cambridge University Press; 1992.
2. Van de Hulst HC. Light scattering by small particles. New York: Dover Publications; 1961.
3. Landau L, Lifschitz L. Electrodynamics of continuous media. Oxford: Pergamon Press; 1984.
4. Ramm AG. Wave scattering by small bodies of arbitrary shapes. Singapore: World Scientific Publishing Co. Pte. Ltd.; 2005.
5. Martin P. Multiple scattering. Cambridge: Cambridge University Press; 2006.
6. Ramm AG. Scattering by obstacles. Dordrecht: D. Reidel; 1986.
7. Ramm AG. Scattering by many small bodies and applications to condensed matter physics. Europhysics Letters 2007; 80(4): 44001. doi: 10.1209/0295-5075/80/44001.
8. Ramm AG. Many-body wave scattering by small bodies and applications. Journal of Mathematical Physics 2007; 48: 103511. doi: 10.1063/1.2799258.
9. Ramm AG. Wave scattering by small particles in a medium. Physics Letters A 2007; 367(1-2): 156–161. doi: 10.1016/j.physleta.2007.02.076.
10. Ramm AG. Wave scattering by small impedance particles in a medium. Physics Letters A 2007; 368(1-2): 164–172. doi: 10.1016/j.physleta.2007.04.061.

11. Ramm AG. Distribution of particles which produces a desired radiation pattern. Communications in Nonlinear Science and Numerical Simulation 2007; 12(7): 1115–1119. doi: 10.1016/j.cnsns.2005.11.001.
12. Ramm AG. Distribution of particles which produces a “smart” material. Journal of Statistical Physics 2007; 127: 915–934. doi: 10.1007/s10955-007-9303-3.
13. Ramm AG. Distribution of particles which produces a desired radiation pattern. Physica B: Condensed Matter 2007; 394(2): 253–255. doi: 10.1016/j.physb.2006.12.019.
14. Ramm AG. Creating wave-focusing materials. Latin American Journal of Solids and Structures 2008; 5(2): 119–127.
15. Ramm AG. Electromagnetic wave scattering by small bodies. Physics Letters A 2008; 372(23): 4298–4306. doi: 10.1016/j.physleta.2008.03.010.
16. Ramm AG. Wave scattering by many small particles embedded in a medium. Physics Letters A 2008; 372(17): 3064–3070. doi: 10.1016/j.physleta.2008.01.006.
17. Ramm AG. Preparing materials with a desired refraction coefficient and applications. In: Skiadas C, Dimotikalis I, Skiadas C (editors). Topics on chaotic systems: Selected papers from Chaos 2008 International Conference. Singapore: World Scientific Publishing Co. Pte. Ltd.; 2009. p. 265–273.
18. Ramm AG. Preparing materials with a desired refraction coefficient. Nonlinear Analysis: Theory, Methods & Applications 2009; 71(12): e186–e190. doi: 10.1016/j.na.2008.10.011.
19. Ramm AG. Creating desired potentials by embedding small inhomogeneities. Journal of Mathematical Physics 2009; 50: 123525. doi: 10.1063/1.3267887.
20. Ramm AG. A method for creating materials with a desired refraction coefficient. International Journal of Modern Physics B 2010; 24(27): 5261–5268. doi: 10.1142/S0217979210056074.
21. Ramm AG. Materials with a desired refraction coefficient can be created by embedding small particles into a given material. International Journal of Structural Changes in Solids 2010; 2(2): 17–23.
22. Ramm AG. Wave scattering by many small bodies and creating materials with a desired refraction coefficient. Afrika Matematika 2011; 22: 33–55. doi: 10.1007/s13370-011-0004-3.
23. Ramm AG. Scattering by many small inhomogeneities and applications. In: Skiadas C, Dimotikalis I, Skiadas C (editors). Topics on chaotic systems: Selected papers from Chaos 2010 International Conference. Singapore: World Scientific Publishing Co. Pte. Ltd.; 2011. p. 41–52.
24. Ramm AG. Collocation method for solving some integral equations of estimation theory. International Journal of Pure and Applied Mathematics 2010; 62(1): 57–65. doi: 10.1504/IJCSM.2009.027874.
25. Ramm AG. Electromagnetic wave scattering by a small impedance particle of arbitrary shape. Optics Communications 2011; 284(16-17): 3872–3877. doi: 10.1016/j.optcom.2011.04.035.
26. Ramm AG. Scattering of scalar waves by many small particles. AIP Advances 2011; 1(2): 022135. doi: 10.1063/1.3600704.
27. Ramm AG. Scattering of electromagnetic waves by many thin cylinders. Results in Physics 2011; 1(1): 13–16. doi: 10.1016/j.rinp.2011.05.002.
28. Ramm AG. Electromagnetic wave scattering by many small perfectly conducting particles of an arbitrary shape. Optics Communications 2012; 285(18): 3679–3683. doi: 10.1016/j.optcom.2012.05.010.
29. Ramm AG. Electromagnetic wave scattering by small impedance particles of an arbitrary shape. Journal of Applied Mathematics and Computing 2013; 43(1): 427–444. doi: 10.1007/s12190-013-0671-3.
30. Ramm AG. Many-body wave scattering problems in the case of small scatterers. Journal of Applied Mathematics and Computing 2013; 41(1): 473–500. doi: 10.1007/s12190-012-0609-1
31. Ramm AG. Scattering of acoustic and electromagnetic waves by small bodies of arbitrary shapes. Applications to creating new engineered materials. New York: Momentum Press; 2013. p. 260.
32. Ramm AG. Creating materials with a desired refraction coefficient. doi: 10.1088/978-1-6817-4708-8. San Rafael, California: IOP Concise Physics, Morgan & Claypool Publishers; 2017.
33. Ramm AG. Creating materials with a desired refraction coefficient. 2nd ed. Bristol, UK: IOP Publishing; 2020. doi: 10.1088/978-0-7503-3391-7.
34. Ramm AG. How can one create a material with a prescribed refraction coefficient? Sun Text Review of Material Science 2020; 1(1): 102. doi: 10.51737/2766-5100.2020.002.
35. Ramm AG. Scattering by obstacles and potentials, Singapore: World Scientific Publishing Co. Pte. Ltd.; 2017. p. 620. doi: 10.1142/10473.
36. Ramm AG, Tran N. A fast algorithm for solving scalar wave scattering problem by billions of particles. Journal of Algorithms and Optimization 2015; 3(1): 1–13.
37. Andriychuk MI, Ramm AG. Numerical solution of many-body wave scattering problem for small particles and creating materials with desired refraction coefficient. In: Awrejcewicz J (editor). Numerical simulations of physical and engineering processes. London: IntechOpen; 2011. p. 1–28. doi: 10.5772/24495.
38. Andriychuk M, Ramm AG. Scattering of electromagnetic waves by many thin cylinders: Theory and computational modeling, Optics Communications 2012; 285(20): 4019–4026. doi: 10.1016/j.optcom.2012.06.017.